[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : **4548** **E**

Unique Paper Code : 32351201

Name of the Paper : Real Analysis (CBCS-LOCF)

Name of the Course : **B.Sc. (Hons) Mathematics**

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any two parts from each question.

1. (a) If x and y are positive real numbers with $x < y$, then prove that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$. (6.5)

   (b) Define Infimum and Supremum of a nonempty set of $\mathbb{R}$. Find infimum and supremum of the set

$$S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}.$$ (6.5)

(c) State the completeness property of $\mathbb{R}$, hence show that every nonempty set of real numbers which is bounded below, has an infimum in $\mathbb{R}$. (6.5)

2. (a) Prove that there does not exist a rational number $r \in \mathbb{Q}$ such that $r^2 = 2$. (6)

(b) Define an open set and a closed set in $\mathbb{R}$. Show that if a, b $\in \mathbb{R}$, then the open interval (a, b) is an open set. (6)

(c) Let S be a nonempty bounded set in $\mathbb{R}$. Let $a > 0$, and let $aS = \{as : s \in S\}$. Prove that inf (aS) = a(inf S) and sup (aS) = a(supS). (6)

3. (a) Define limit of a sequence. Using definition show that $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$. (6.5)

(b) Prove that every convergent sequence is bounded. Is the converse true? Justify. (6.5)

(c) Let $x_1 = 1$ and $x_{n+1} = \frac{1}{4}(2x_n + 3)$ for $n \in \mathbb{N}$. Show that $\langle x_n \rangle$ is bounded and monotone. Find the limit. (6.5)

4\. (a) If $\langle a_n$ ... nd $\langle b_n \rangle$ converges to a and b respect... , prove that $\langle a_n b_n \rangle$ converges to ab. (6)

(b) Show th... $\lim_{n\to\infty} n^{1/n} = 1$. (6)

(c) State ... chy Convergence Criterion for sequen... Hence show that the sequence $\langle a_n \rangle$, define... y $a_n = 1 + \frac{1}{2} + \cdots\cdots + \frac{1}{n}$, does not conver... (6)

5\. (a) Prove ... if an infinite series $\sum_{n=1}^{\infty} a_n$ is conve... then $\lim_{n\to\infty} a_n = 0$. Hence examine the conv... of $\sum_{n=1}^{\infty} \frac{n}{2n+3}$. (6.5)

(b) Exam... e convergence or divergence of the follow... ries.

(i) ... $\frac{4}{3} + \frac{6}{11} + \cdots$ (6.5)

P.T.O.

(ii) $\sum_{n=1}^{\infty}\left(\frac{3n+5}{2n+1}\right)^{n/2}$

(c) Prove that $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{p}}, p>0$ is convergent for $p>1$ and divergent for $p\leq 1$. (6.5)

6. (a) State and prove ratio test (limit form). (6)

(b) Examine the convergence or divergence of the following series. (6)

(i) $\sum_{n=1}^{\infty}\frac{n^{3}+1}{n^{4}+3n^{2}+2n}$

(ii) $3+\frac{3^{2}}{2!}+\frac{3^{3}}{3!}+\frac{3^{4}}{4!}+\cdots$

(c) Prove that the series $\frac{1}{\ln 2}-\frac{1}{\ln 3}+\frac{1}{\ln 4}-\frac{1}{\ln 5}+\cdots$ is conditionally convergent. (6)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : **4530** **E**

Unique Paper Code : 32351401

Name of the Paper : BMATH408- Partial Differential Equations

Name of the Course : **B.Sc.(H) Mathematics**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All sections are compulsory.
3. Marks of each part are indicated.

Section – I

1. Attempt any two out of the following: [7.5+7.5]

(a) Find the integral surfaces of the equation $u\,u_x + u_y = 1$ for the initial data:

$$x(s,0) = s, y(s,0) = 2s, u(s,0) = s.$$

(b) Apply $\sqrt{u} = v$ and $v(x,y) = f(x) + g(y)$ to solve:

$$x^4\,u_x^2 + y^2\,u_y^2 = 4\,u.$$

(c) Find the solution of the initial–value systems

$$u_t + u\,u_x = e^{-x}v,\ v_t - av_x = 0,$$

with $u(x,0) = x$ and $v(x,0) = e^x$.

Section – II

2. Attempt any one out of the following: [6]

(a) Derive the two- dimensional wave equation of the vibrating membrane

$$u_{tt} = c^2(u_{xx} + u_{yy}) + F,$$

where, $c^2 = T/\rho$, and T is the tensile force per unit length

$F = f/\rho$, and f be the external force, acting on the membrane.

(b) Drive the potential equation $\nabla^2 V = 0$, where ∇^2 is known as Laplace operator.

3. Attempt any two out of the following: [6+6]

(a) Determine the general solution of

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2.$$

(b) Given that the parabolic equation

$$u_{xx} = a\,u_t + b\,u_x + c\,u + f,$$

where the coefficients are constants, by the substitution $u = v\,e^{\frac{1}{2}bx}$ and for the case $c = -(b^2/4)$, show that the given equation is reduced to the heat equation

$$v_{xx} = a\,v_t + g,$$

where $g = f\,e^{-bx/2}$.

(c) Reduce the equation

$$(n-1)^2 u_{xx} - y^{2n}\,u_{yy} = n\,y^{2n-1}\,u_y,$$

to canonical form for $n = 1$ and $n = 2$ if possible and also find their solutions.

Section – III

4. Attempt any three parts out of the following: [7+7+7]

(a) Determine the solution of the given below initial-value problem

$$u_{tt} - c^2 u_{xx} = x, \qquad u(x,0) = 0, \quad u_t(x,0) = 3.$$

(b) Obtain the solution of the initial boundary-value problem

$u_{tt} = 9u_{xx}, \quad 0 < x < \infty, \ t > 0,$

$u(x, 0) = 0, \quad 0 \leq x < \infty,$

$u_t(x, 0) = x^3, \quad 0 \leq x < \infty,$

$u_x(0, t) = 0, \quad t \geq 0.$

(c) Solve:

$u_{tt} = c^2 u_{xx},$

$u(x, t) = f(x)$ on $t = t(x),$

$u(x, t) = g(x)$ on $x + ct = 0,$

where $f(0) = g(0)$.

(d) Determine the solution of the initial boundary-value problem:

$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \ t > 0$

$u(x, 0) = f(x), \quad 0 \leq x \leq l,$

$u_t(x, 0) = g(x), \quad 0 \leq x \leq l,$

$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0.$

Section -- IV

5. Attempt any three out of the following: [7+7+7]

(a) Determine the solution of the initial boundary value problem:

$u_t = 4\, u_{xx}, \quad 0 < x < 1, \ t > 0$

$u(x, 0) = x^2\,(1 - x), \quad 0 \leq x \leq 1,$

$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0.$

(b) Determine the solution of the initial boundary value problem by the method of separation of variables:

$u_{tt} = c^2 u_{xx}, \quad 0 < x < \pi, \ t > 0$

$u(x, 0) = 0, \quad 0 \leq x \leq \pi,$

$u_t(x, 0) = 8 \sin^2 x, \quad 0 \leq x \leq \pi,$

$u(0, t) = 0, \quad u(\pi, t) = 0, \quad t \geq 0.$

(c) Solve by using method of separation of variables:

$u_{tt} - u_{xx} = h,$ $\quad 0 < x < 1,\ t > 0$, h is a constant

$u(x, 0) = x^2,$ $\quad 0 \le x \le 1,$

$u_t(x, 0) = 0,$ $\quad 0 \le x \le 1,$

$u(0, t) = 0, \quad u(1, t) = 0, \quad t \ge 0.$

(d) State and prove the uniqueness of solution of the heat conduction problem.

[This question paper contains 2 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : **4686** **E**

Unique Paper Code : 32351402

Name of the Paper : Riemann Integration and Series of Functions

Name of the Course : **B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **two** parts from each question.

1(a) Let f be integrable on [a, b], and suppose g is a function on [a, b] such that $g(x) = f(x)$ except for finitely many x in [a, b]. Show g is integrable and $\int_a^b g = \int_a^b f$ (6)

(b) Show that if f is integrable on [a, b] then f^2 also is integrable on [a, b]. (6)

(c) (i) Let f be a continuous function on [a,b] such that $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x)\, dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$ (3)

(ii) Give an example of function such that $|f|$ is integrable on $[0,1]$ but f is not integrable on $[0,1]$. Justify it. (3)

2(a) State and prove Fundamental Theorem of Calculus I. (6.5)

(b) State Intermediate Value Theorem for Integrals. Evaluate $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\, dt$. (6.5)

(c) Let function $f: [0,1] \to R$ be defined as

$$f(x) = \begin{cases} x^2 & if\ x\ is\ \ rational \\ 0 & if\ x\ is\ \ irrational \end{cases}$$

Calculate the upper and lower Darboux integrals for f on the interval [0,1]. Is f integrable on [0.1] ? (6.5)

3(a) Examine the convergence of the improper integral $\int_0^\infty e^{-x}x^{n-1}dx$. (6)

(b) Show that the improper integral $\int_\pi^\infty \frac{Sinx}{x}\, dx$ is convergent but not absolutely convergent. (6)

P.T.O.

(c) Determine the convergence or divergence of the improper integral (6)

(i) $\int_0^1 \frac{dx}{x(lnx)^2}$

(ii) $\int_1^\infty \frac{xdx}{\sqrt{x^3+x}}$

4(a) Show that the sequence

$$f_n(x) = \frac{nx}{1+nx}, \ x \in [0,1], \ n \in N$$

converges non-uniformly to an integrable function f on [0,1] such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx \quad (6.5)$$

(b) Show that the sequence $\{x^2 e^{-nx}\}$ converges uniformly on $[0, \infty)$. (6.5)

(c) Let $\langle f_n \rangle$ be a sequence of continuous function on $A \subset R$ and suppose that $\langle f_n \rangle$ converges uniformly on A to a function $f: A \longrightarrow R$. Show that f is continuous on A. (6.5)

5(a) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \geq 0$. Show that sequence $\langle f_n \rangle$ converges non-uniformly on $[0, \infty)$ and converges uniformly on $[a, \infty), a > 0$. (6.5)

(b) State and prove Weierstrass M-test for the uniform Convergence of a series of functions. (6.5)

(c) Show that the series of functions $\sum \frac{1}{n^2+x^2}$, converges uniformly *on* R to a continuous function. (6.5)

6(a) (i) Find the exact interval of convergence of the power series (3)

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n}$$

(ii) Define $sinx$ as a power series and find its radius of convergence (3)

(b) Prove that $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$ for $|x| < 1$ and hence evaluate $\frac{\sum_{n=1}^{\infty} n^2(-1)^n}{3^n}$. (6)

(c) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad for \ |x| < R. \quad (6)$$

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4810 **E**

Unique Paper Code : 32351403

Name of the Paper : Ring Theory & Linear Algebra – I

Name of the Course : **B.Sc. [Hons.] Mathematics CBCS (LOCF)**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts from each question.

P.T.O.

1. (a) Find all the zero divisors and units in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$. (6)

(b) Prove that characteristic of an integral domain is 0 or prime number p. (6)

(c) State and prove the Subring test (6)

2. (a) Let R be a commutative ring with unity and let A be an ideal of R then prove that R/A is a field if and only if A is a maximal ideal of R. (6)

(b) Let A and B are two ideals of a commutative ring R with unity and A+B=R then show that $A \cap B = AB$. (6)

(c) If an ideal I of a ring R contains a unit then show that I=R. Hence prove that the only ideals of a field F are {0} and F itself. (6)

3. (a) Find all ring homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_{15}$. (6.5)

(b) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$ and Φ be the mapping

that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to a-b. Show that

(i) Φ is a ring homomorphism.

(ii) Determine Ker Φ.

(iii) Show that R/Ker Φ is isomorphic to $\mathbb{Z}$. (6.5)

(c) Using homomorphism, prove that an integer n with decimal representation $a_k \, a_{k-1} \ldots \ldots a_0$ is divisible by 9 iff $a_k + a_{k-1} + \ldots + a_0$ is divisible by 9.

(6.5)

4. (a) Let V(F) be the vector space of all real valued function over $\mathbb{R}$.

Let $V_e = \{f \in V \mid f(x) = f(-x) \ \forall \ x \in \mathbb{R})$

and $V_o = \{f \in V \mid f(-x) = -f(x) \ \forall \ x \in \mathbb{R})$

Prove that V_e and V_0 are subspaces of V and $V = V_e \oplus V_o$. (6)

(b) Let V(F) be a vector space and let $S_1 \subseteq S_2 \subseteq V$. Prove that

(i) If S_1 is linearly dependent then S_2 is linearly dependent

(ii) If S_2 is linearly independent then S_1 is linearly independent (6)

(c) Show that $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ forms a basis for $M_{2X2}(\mathbb{R})$. (6)

5. (a) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

Find Null space and Range space of T and verify Dimension Theorem. (6.5)

(b) Define $T: M_{2X2}(\mathbb{R}) \to P_2(\mathbb{R})$ by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$

Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\gamma = \{1, x, x^2\}$ be basis of $M_{2X2}(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. Compute $[T]_\beta^\gamma$. (6.5)

(c) Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. For $w_1, w_2, \ldots, w_n$ in W. Prove that there exists exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$. (6.5)

6. (a) Let T be the linear operator on $\mathbb{R}^2$ define by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$$

Let β be the standard ordered basis for $\mathbb{R}^2$ and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Find $[T]_{\beta'}$. (6.5)

(b) Let V and W be finite dimensional vector spaces with ordered basis β and γ respectively. Let $T: V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible.

Furthermore, $\left[T^{-1}\right]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$. (6.5)

(c) Let V, W and Z be finite dimensional vector spaces with ordered basis α, β, γ respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformations.

Then $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}$. (6.5)

(1000)

[This question paper contains 8 printed pages.]

Your Roll No.............

Sr. No. of Question Paper : 4512

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt **two** parts from each question.

1\. (a) Find and sketch, showing corresponding orientations, the images of the hyperbolas

$x^2 - y^2 = c_1$ $(c_1 < 0)$ and $2xy = c_2$ $(c_2 < 0)$

under the transformation $w = z^2$. (6)

(b) (i) Prove that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2$$

as z tends to 0 does not exist.

(ii) Show that

$$\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4.$$

(3+3=6)

(c) Show that the following functions are nowhere differentiable.

(i) $f(z) = z - \bar{z}$,

(ii) $f(z) = e^y \cos x + i e^y \sin x$. (3+3=6)

(d) (i) If a function f(z) is continuous and nonzero at a point z_0, then show that $f(z) \neq 0$ throughout some neighborhood of that point.

(ii) Show that the function $f(z) = (z^2 - 2)e^{-x}e^{-iy}$ is entire. (3+3=6)

2. (a) (i) Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$|\exp \exp(2z + i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}$.

(ii) Find the value of z such that

$$e^z = 1 + \sqrt{3}\,i \qquad (3.5+3=6.5)$$

(b) Show that

(i) $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z;

(ii) $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = n\pi i$ $(n = 0 \pm 1, \pm 2, ...)$. (3.5+3=6.5)

(c) Show that

(i) log log (i^2) = 2logi where

$$\log z = \text{In}r + i\theta (r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}).$$

(ii) log log $(i^2) \neq$ 2logi where

$$\log z = \text{In}r + i\theta (r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}).$$

(3.5+3=6.5)

(d) Find all zeros of sinz and cosz. (3.5+3=6.5)

3. (a) State Fundamental theorem of Calculus.

Evaluate the following integrals to test if Fundamental theorem of Calculus holds true or not :

(i) $\int_0^{\pi/2} \exp(t + it)dt$

(ii) $\int_0^1 (3t - i)^2 dt$ (2+2+2=6)

(b) Let y(x) be a real valued function defined piecewise on the interval $0 \le x \le 1$ as

$y(x) = x^3 \sin(\pi/x)$, $0 < x \le 1$ and $y(0) = 0$

Does this equation $z = x + iy$, $0 \le x \le 1$ represent

(i) an arc

(ii) A smooth arc. Justify.

Find the points of intersection of this arc with real axis. (2+2+2=6)

(c) For an arbitrary smooth curve C: $z = z(t)$, $a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integral depends only on the end points of C.

State if it is independent of the arc under consideration or not?

Also, find its value around any closed contour.

(3+1+2=6)

(d) Without evaluation of the integral, prove that

$\left|\int_c \frac{1}{z^2+1} dz\right| \le \frac{1}{2\sqrt{5}}$ where C is the straight line segment from 2 to 2 +i. Also, state the theorem used.

(4+2=6)

4. (a) Use the method of antiderivative to show that

$\int_c (z-z_0)^{n-1} dz = 0$, n = ±1, ±2,-- where C is any closed contour which does not pass through the point z_0. State the corresponding result used.

(4+2.5=6.5)

(b) Use Cauchy Gourset theorem to evaluate :

(i) $\int_c f(z)dz$, when $f(z) = \frac{1}{z^2+2z+2}$ and C is the unit circle |z| = 1 in either direction.

(ii) $\int_c f(z)dz$, when $f(z)=\frac{5z+7}{z^2+2z-3}$ and C is the circle $|z-2| = 2$. (3+3.5=6.5)

(c) State and prove Cauchy Integral Formula. (2+4.5=6.5)

(d) Evaluate the following integrals :

(i) $\int_c \frac{\cos z}{z(z^2+8)}dz$, where C is the positive oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

(ii) $\int_c \frac{2s^2-s-2}{s-2}ds$, $|z| \neq 3$ at $z = 2$, where C is the circle $|z| = 3$. (3.5+3=6.5)

5. (a) If a series of complex numbers converges then prove that the nth term converges to zero as n tends to infinity. Is the converse true? Justify. (6.5)

(b) Find the Maclaurin series for the function $f(z) = \sinh z$. (6.5)

(c) If a series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges to f(z) at all points interior to some circle $|z - z_0| = R$, then prove that it is the Taylor series for the function f(z) in powers of $z - z_0$. (6.5)

(d) Find the integral of f(z) around the positively oriented circle $|z| = 3$ when $f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$. (6.5)

6. (a) For the given function $f(z) = \left(\frac{z}{2z+1}\right)^3$, show any singular point is a pole. Determine the order of each pole and find the corresponding residue. (6)

(b) Find the Laurent Series that represents the function $f(z) = z^2 \sin\frac{1}{z^2}$ in the domain $0 < |z| < \infty$. (6)

(c) Suppose that $z_n = x_n + iy_n$, $(n = 1,2,3,...)$ and $S = X + iY$. Then show that

$$\sum_{n=1}^{\infty} z_n = S \text{ iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

(6)

(d) If a function $f(z)$ is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then show that

$$\int_c f(z)\,dz = 2\pi i\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

(6)

(20)

[This question paper contains 4 printed pages.]

Your Roll No. ....

Sr. No. of Question Paper : 4792 **E**

Unique Paper Code : 32351602

Name of the Paper : Ring Theory and Linear Algebra – II

Name of the Course : **B.Sc. (H) Mathematics (CBCS – LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** the questions are compulsory.
3. Attempt any **two** parts from each question.
4. Marks of each part are indicated

1. (a) (i) Prove that If F is a field, then F[x] is a Principal Ideal Domain.

   (ii) Is $\mathbb{Z}[x]$, a Principal Ideal Domain? Justify your answer.

   (b) Prove that $<x^2 + 1>$ is not a maximal ideal in $\mathbb{Z}[x]$.

(c) State and prove the reducibility test for polynomials of degree 2 or 3. Does it fail in higher order polynomials? Justify. (4+2,6,6)

2. (a) (i) State and prove Gauss's Lemma.

(ii) Is every irreducible polynomial over $\mathbb{Z}$ primitive? Justify.

(b) Construct a field of order 25.

(c) In $\mathbb{Z}\left[\sqrt{(-5)}\right]$, prove that $1+3\sqrt{(-5)}$ is irreducible but not prime. (4+2.5,6.5,6.5)

3. (a) Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows :

$f_1(x, y, z) = x - 2y,$

$f_2(x, y, z) = x + y + z,$

$f_3(x, y, z) = y - 3z.$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^* and then find a basis for V for which it is the dual basis.

(b) Test the linear operator $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f(0) + f(1)(x + x^2)$ for diagonalizability and if diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(c) Let $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix} \in M_{2\times2}(\mathbb{R})$. Find an expression for A^n where n is an arbitrary natural number. (6,6,6)

4. (a) For a linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(a, b, c) = (-b + c, a + c, 3c)$, determine the T-cyclic subspace W of $\mathbb{R}^3$ generated by $e_1 = (1, 0, 0)$. Also find the characteristic polynomial of the operator T_W.

(b) State Cayley-Hamilton theorem and verify it for the linear operator $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f'(x)$.

(c) Show that the vector space $\mathbb{R}^4 = W_1 \oplus W_2 \oplus W_3$ where $W_1 = \{(a, b, 0, 0): a, b \in \mathbb{R}\}$, $W_2 = \{(0,0, c, 0): c \in \mathbb{R}\}$ and $W_3 = \{(0,0,0, d): d \in \mathbb{R}\}$.

(6.5,6.5,6.5)

5. (a) Consider the vector space $\mathbb{C}$ over $\mathbb{R}$ with an inner product $\langle .,. \rangle$. Let $\overline{z}$ denote the conjugate of z. Show that $\langle .,. \rangle'$ defined by $\langle z, w \rangle' = \langle \overline{z}, \overline{w} \rangle$ for all $z, w \in \mathbb{C}$ is also an inner product on $\mathbb{C}$. Is $\langle .,. \rangle''$ defined by $\langle z, w \rangle'' = \langle z + \overline{z}, w + \overline{w} \rangle$ for all $z, w \in \mathbb{C}$ an inner product on $\mathbb{C}$? Justify your answer.

(b) Let $V = P(\mathbb{R})$ with the inner product $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(t)q(t)\,dt \;\; \forall p(x), q(x) \in V$. Compute the orthogonal projection of the vector $p(x) = x^{2k-1}$ on $P_2(\mathbb{R})$, where $k \in \mathbb{N}$.

(c) (i) For the inner product space $V = P_1(R)$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$ and the linear operator T on V defined by $T(f) = f' + 3f$, compute $T^*(4 - 2t)$.

(ii) For the standard inner product space $V = \mathbb{R}^3$ and a linear transformation $g: V \to \mathbb{R}$ given by $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$, find a vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$.

(6,6,2+4)

6. (a) Prove that a normal operator T on a finite-dimensional complex inner product space V yields an orthonormal basis for V consisting of eigenvectors of T. Justify the validity of the conclusion of this result if V is a finite-dimensional real inner product space.

(b) Let $V = M_{2\times 2}(\mathbb{R})$ and $T: V \to V$ be a linear operator given by $T(A) = A^T$. Determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(c) For the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$.

(6.5,6.5,6.5)

[This question paper contains 12 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4748 **E**

Unique Paper Code : 32357609

Name of the Paper : DSE-3 : Bio-Mathematics

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. Use of Scientific Calculator is allowed.

1. (a) A doctor has to prescribe medicine to the patient. The medicine raises the blood plasma concentration of an average adult by 20 mg l^{-1} and takes 6 h to decay in the blood plasma. The maximum

P.T.O.

permissible limit of concentration of drug in the body is 40 mg l^{-1}. What time gap he will ensure for maintaining a safe decomposition of drug. Find out the minimum concentration if doses are given at this time interval? The concentration of another drug, decreases by 40% in 20 h. Find how long will it take for this drug to fall to 5% of its initial value. (6)

(b) Observation on animal tumours indicate that theirsizes obey the Gompertz growth law $\frac{ds}{dt} = ks\ln\left(\frac{S}{s}\right)$ rather than the logistic law. Here k and S are positive constants. By putting y = ln(s), prove that s(t) = $Se^{-Ae^{-kt}}$, where $A = \ln\left(\frac{S}{s_0}\right)$, s_0

being the size at $t = 0$. Discuss the model describing drug concentration and residual concentration at any time nto, in which drug decays according to equation $\frac{dc}{dt} = \frac{-c(t)}{\tau}$. When dose is administered regularly at time $t = 0, t_0, 2t_0, 3t_0, \ldots$ with assumption that each dose raises the drug concentration by fixed amount Co. Find the maximum possible concentration and residue as n increases. (6)

(c) Consider the following chemical reaction, with the rate constant as q :

$$P + X \rightarrow 7Y.$$

If the reactant P is held at a constant concentration p, derive a system of equations for the concentrations of X and Y. Suppose the initial concentrations of X and Y are X_0 and Y_0 respectively. Solve the system of equations to obtain X(t) and Y(t). (6)

(d) Show that Zeeman's heartbeat equations have a unique resting state

$$x = x_a,\ b = -(x_a^3 + a\,x_a).$$

Then derive the single differential equation satisfied by muscle fibre of length x. (6)

2. (a) Discuss the nature of fixed point and give equation of trajectories for the given system

$$\dot{x} = 6x + 12y$$
$$\dot{y} = 3x + y$$ (6½)

(b) (i) Discuss the two equilibrium states during the heart beat cycle and the role of pacemaker in the heartbeat cycle.

(ii) Discuss the threshold level and firing of axon in nerve impulse transmission. (6½)

(c) Examine the possibility of periodic solutions of

$$c\ddot{x} + (2 + 3ax + 4bx^2)x = 0$$

Where a, b and c are constants, c being positive. (6½)

(d) Describe the epidemic model and show that population return to equilibrium after the small departure from the equilibrium. (6½)

3. (a) Consider the system

$$\frac{du}{dt} = u\ (1 - u)\ (u - a) - w$$

$$\frac{dw}{dt} = bu - \gamma w.$$

Where $0 < a < 1$, $b > 0$, $\gamma \geq 0$.

Linearise the above system about (0,0). Further assuming that $u = \alpha e^{\lambda t}$, $w = \beta e^{\gamma t}$ be solutions of the linearised system about (0,0), show that the rest state (0,0) is locally stable. (6)

(b) Sketch the trajectories of the following system

$$\dot{x} = y$$

$$\dot{y} = \frac{1}{2}(1 - x^2) \qquad (6)$$

(c) Define

(i) Bifurcation

(ii) Bifurcation Point

Make the sketches for Pitchfork bifurcation, Saddle-node bifurcation and Hopf bifurcation.

(6)

(d) For the iteration scheme $x_{n+1} = \mu x_n(1 - x_n)$, $n \geq 1$,

$$x_0 = \lim_{n\to\infty} x_n$$

Show that there are bifurcations at $\mu = 1$ and $\mu = 3$. (6)

4. (a) Find the constraints on a, b and λ assuming it has a unique rest state, taking the solutions to the travelling wave equations in the form $u = \phi(x + ct)$, $w = \psi(x + ct)$ of the following system of equations $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a) - w$,

$\frac{\partial w}{\partial t} = bu - \gamma w.$ (6½)

(b) What is Flopf bifurcation? Show that Hopf bifurcation holds for the following system

$$\dot{x} = -y + x(\mu - x^2 - y^2)$$

$$\dot{y} = x + y(\mu - x^2 - y^2)$$ (6½)

(c) Provide a full phase plane analysis for the mathematical model of heart beat equations given by

$$\varepsilon\frac{dx}{dt} = -\left(x^3 - Tx + b\right), \qquad T > 0$$

$$\frac{db}{dt} = x - x_0.$$

Where x is the muscle fibre length, b is the chemical control, $\varepsilon > 0$ and (x_0, b_0) is a rest state. (6½)

(d) Show that the following system has limit cycle.

$$\frac{du}{dt} = u(1-u)(u-a) - w + I(t),$$

$$\frac{dw}{dt} = bu - \gamma w,\ 0 < a < 1,\ b > 0,\ \gamma \geq 0. \qquad (6½)$$

5. (a) Write down the steps of Neighbor Joining Algorithm. From the given distance table of four taxa S_1, S_2, S_3 and S_4, compute R_1, R_2, R_3, R_4 and then form a table of values for $M(S_i, S_j)$ for $1 \leq i \neq j \leq 4$.

	S_1	S_2	S_3	S_4
S_1		1.2	0.9	1.7
S_2			1.1	1.9
S_3				1.6

(6½)

(b) If D and d denote the alleles for tall and dwarf plant and if W and w denote the alleles for round and wrinkled seed, then create a Punnett square for a DdWw × ddWw cross pea plant and compute the probability of a tall plant with wrinkled seeds. (6½)

(c) Derive the formula for the Jukes-Cantor distance (d_{JC}) given that all the diagonal entries of Jukes Cantor matrix M^t are $\frac{1}{4}+\frac{3}{4}\left(1-\frac{4}{3}\alpha\right)^t$, where a is the mutation rate. Compute the Jukes-Cantor distance $d_{JC}(S_0,S_1)$ to 4 decimal digits, from the following 40 base table : (6½)

$S_1 \backslash S_0$	A	G	C	T
A	7	0	1	1
G	1	9	2	0
C	0	2	7	2
T	1	0	1	6

(d) Describe when two trees are considered to be topologically similar. Draw all topologically distinct un-rooted bifurcation trees that could describe the relationship between 3 taxa and 4 taxa. (6½)

6. (a) Define phylogenetic tree, bifurcating tree and unrooted tree with examples of each. (6)

(b) Explain Kimura 2-parameter and 3-parameter models along with their corresponding distance formulas. Write the expression of the log-det distance between S_0 and S_1. (6)

(c) In mice, an allele A for agouti- or gray-brown grizzled fur is dominant over the allele a, which determines a non-agouti color. If an Aa×Aa cross produces 4 offsprings, then compute the probabilities that :

(i) No offspring have agouti fur.

(ii) Exactly 3 of 4 offspring have agouti fur.

(6)

(d) From the given distance table of four sequences S_1, S_2, S_3 and S_4 of DNA, construct a rooted tree showing the relationship between S_1, S_2, S_3 and S_4 by UPGMA

	S_1	S_2	S_3	S_4
S_1		0.45	0.27	0.53
S_2			0.40	0.50
S_3				0.62

(6)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4873 **E**

Unique Paper Code : 32357610

Name of the Paper : DSE-4 (Number Theory)

Name of the Course : **CBCS (LOCF) – B.Sc. (H) (Mathematics)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts of each question.
4. Question Nos. **1** to **3**, each part carries **6.5** marks and Question Nos. **4** to **6**, each part carries **6** marks.

1. (a) Determine all solutions in the positive integers of the Diophantine equation

$$18x + 5y = 48.$$

P.T.O.

(b) Using Euclidean algorithm and theory of linear Diophantine equation, divide 100 into two summands such that one is divisible by 7 and other by 11.

(c) Write a short note on Prime number theorem.

(d) If $ca \equiv cb \pmod{n}$ then prove that $a \equiv b \pmod{n/d}$, where $d = \gcd(c, n)$.

2. (a) Verify that $0, 1, 2, 2^2, 2^3 \ldots\ldots 2^9$ form a complete set of residues modulo 11, but that $0, 1^2, 2^2, 3^2 \ldots\ldots 10^2$ do not.

(b) Find the solutions of the system of congruences :

$$3x + 4y \equiv 5 \pmod{13}$$

$$2x + 5y \equiv 7 \pmod{13}.$$

(c) Use Fermat's theorem to verify that 17 divides $11^{104} + 1$.

(d) Find the remainder when $2(26!)$ is divided by 29.

3. (a) Let F and f be two number – theoretic functions related by the formula

$$F(n)=\sum_{d/n} f(d)$$

Prove $f(n) = \Sigma_{d|n}\mu(d)F(n/d) = \Sigma_{d|n}\mu(n/d)F(d)$.

(b) Verify that 1000! terminates in 249 zeros.

(c) Use Euler's theorem for any integer a, to prove that $a^{13} \equiv a \pmod{2730}$

(d) Prove that $\varnothing(2^n - 1)$ is a multiple of n for any $n > 1$.

4. (a) For any positive integer n, prove

$\varnothing(n) = n\Sigma_{d|n}\mu(d)/d$.

(b) Define primitive roots of an integer by an example and show that if $F_n = 2^{2^n} + 1$, $n > 1$, is a prime then 2 is not a primitive root of F_n.

(c) If p is a prime number and $d|p-1$, then show that there are exactly $\varphi(d)$ incongruent integers having order d modulo p.

(d) Determine all the primitive roots of the primes $p = 11, 19$, and 23, expressing each as a power of one of the roots.

P.T.O.

5. (a) If gcd (m, n) = 1, where $m > 2$ and $n > 2$, then prove that the integer 'mn' has no primitive roots.

(b) Solve the quadratic congruence

$3x^2 + 9x + 7 \equiv 0 \pmod{13}$.

(c) Show that 3 is quadratic residue of 23, but a nonresidue of 31.

(d) Prove that there are infinitely many primes of the form 4k+1.

6. (a) Find the value of Legendre symbol (1234/4567).

(b) Solve the quadratic congruence

$x^2 \equiv 23 \pmod{7^3}$.

(c) Using the linear cipher $C \equiv 5P + 11 \pmod{26}$, encrypt the message NUMBER THEORY IS EASY.

(d) When the RSA algorithm is based on the key (n, k) = (3233, 37), what is the recovery exponent for the cryptosystem?

[This question paper contains 8 printed pages.]

Your Roll No.............

Sr. No. of Question Paper : 4874 **E**

Unique Paper Code : 32357616

Name of the Paper : DSE-4 Linear Programming and Applications

Name of the Course : **CBCS (LOCF)– B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

3. **All** questions carry equal marks.

1. (a) Solve the following Linear Programming Problem by Graphical Method :

Maximize $2x + y$

subject to $x + 2y \leq 10$

$x + y \leq 6$

$x - y \leq 2$

$x - 2y \leq 1$

$x \geq 0, y \geq 0.$

(b) Define a Convex Set. Show that the set S defined as :

$S = \{(x, y) \mid y^2 \geq 4ax;\ x \geq 0,\ y \geq 0\}$ is a Convex Set.

(c) Find all basic feasible solutions of the equations :

$x_1 + 2x_2 + 4x_3 + x_4 = 7$

$2x_1 - x_2 + 3x_3 - 2x_4 = 4$

(d) Prove that to every extreme point of the feasible region, there corresponds a basic feasible solution of the Linear Programming Problem :

Minimize $z = cx$

subject to $Ax = b,\ x \geq 0$

2. (a) Let us consider the following Linear Programming Problem :

Minimize $z = cx$

subject to $Ax = b$, $x \geq 0$

Let $(x_B,0)$ be a basic feasible solution corresponding to a basis B where $x_B = B^{-1}b$. Suppose $z_0 = c_B x_B$ is the value of objective function such that $z_j - c_j \leq 0$ for every column a_j in A. Show that z_0 is the minimum value of z of the problem and that the given basic feasible solution is optimal feasible solution.

(b) Let $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ be a feasible solution to the system of equations :

$$x_1 + x_2 + 2x_3 = 4$$
$$2x_1 - x_2 + x_3 = 2$$

Is this a basic feasible solution? If not, reduce it to two different basic feasible solutions.

(c) Using Simplex Method, find the solution of the following Linear Programming Problem :

Maximize $5x_1 + 4x_2$

subject to $x_1 + 2x_2 \leq 6$

$-x_1 + x_2 \leq 4$

$5x_1 + x_2 \leq 15$

$x_1, x_2 \geq 0.$

(d) Solve the following Linear Programming Problem by Big - M Method :

Maximize $-x_1 - x_2 + x_3$

subject to $x_1 - x_2 - x_3 = 1$

$-x_1 + x_2 + 2x_3 \geq 1$

$x_1, x_2, x_3 \geq 0.$

3. (a) Solve the following Linear Programming Problem by Two Phase Method :

Maximize $-x_1 + 3x_2$

subject to $x_1 + x_2 \geq 1$

$-2x_1 + 3x_2 \leq 6$

$x_1 \geq 0, x_2 \geq 0.$

(b) Find the solution of given system of equations using Simplex Method :

$$5x_1 + 2x_2 = 14$$

$$2x_1 + x_2 = 6$$

Also find the inverse of A where $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$.

(c) Find the solution of the following Linear Programming Problem :

Minimize $3x_1 + 2x_2$

subject to $-x_1 + x_2 \le 1$

$5x_1 + 3x_2 \le 15$

$x_1 \ge 0,\ x_2 \ge 3/2.$

(d) Find the optimal solution of the Assignment Problem with the following cost matrix :

Job \ Machines	I	II	III	IV	V	VI
A	3	4	6	5	4	9
B	5	4	9	7	6	10
C	8	7	6	5	4	6
D	7	4	5	11	10	4
E	5	6	7	8	5	9
F	4	3	5	6	7	4

4. (a) Find the dual of the following Linear Programming Problem :

Maximize $3x_1 + 4x_2 - 3x_3$

Subject to $x_1 - 2x_2 + 5x_3 \geq 2$

$3x_1 + 7x_2 - 4x_3 = -8$

$x_1 \leq 0,\ x_2 \geq 0,\ x_3$ is unrestricted.

(b) Prove that if the Primal Problem has a finite optimal solution then the Dual also has a finite optimal solution and the two optimal objective function values are equal.

(c) Using Complementary Slackness Theorem, find optimal solutions of the following Linear Programming Problem and its Dual.

Minimize $2x_1 + 15x_2 + 5x_3 + 6x_4$

subject to $x_1 + 6x_2 + 3x_3 + x_4 \geq 2$

$-2x_1 + 5x_2 - x_3 + 3x_4 \leq -3$

$x_1, x_2, x_3, x_4 \geq 0.$

(d) For the following cost minimization Transportation Problem find initial basic feasible solutions by using North West Comer rule, Least Cost Method and Vogel's Approximation Method. Compare the three solutions (in terms of the cost) :

Source \ Destination	A	B	C	D	E	Supply
I	10	10	11	12	10	20
II	13	14	11	15	10	35
III	11	10	17	12	15	40
Demand	25	10	30	15	15	

5. (a) Solve the following cost minimization Transportation Problem :

Origin \ Destinations	I	II	III	IV	Availability
A	14	11	13	12	22
B	13	17	10	15	15
C	13	15	16	14	8
Requirements	7	12	17	9	

(b) A University wish to allocate four subjects and six teachers claim that they have the required knowledge to teach all the subjects. Each subject can be assigned to one and only one teacher. The cost of Assignment of subject to each teacher is given in table below. Allocate the subjects to appropriate faculty members for optimal Assignment. Also find two Teachers who are not assigned any course.

Subjects / Teachers	A	B	C	D
I	28	47	36	38
II	36	43	43	46
III	43	38	36	33
IV	47	48	31	38
V	48	38	41	43
VI	43	52	43	49

(c) Define rectangular fair Game. Using Maxmin and Minmax Principle, find the maximum pay-off for player 1 will have and minimum pay-off for player 2 for the following pay-off matrix :

$$\begin{array}{c} \text{Player 1} \\ \text{Player 2} \begin{bmatrix} 10 & 8 & 4 \\ 9 & -5 & 15 \\ -1 & 7 & 6 \end{bmatrix}. \end{array}$$

(d) Convert the following Game Problem into a Linear Programming Problem for player A and player B and solve it by Simplex Method :

$$\begin{array}{c} \text{Player B} \\ \text{Player A} \begin{bmatrix} 6 & 3 & 5 \\ 2 & 5 & 1 \end{bmatrix}. \end{array}$$

(500)

B.Sc (H) mathematics II, IV & VI Sem. May June 2023

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4873 **E**

Unique Paper Code : 32357610

Name of the Paper : DSE-4 (Number Theory)

Name of the Course : **CBCS (LOCF) – B.Sc. (H) (Mathematics)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

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P.T.O.

(b) Using Euclidean algorithm and theory of linear Diophantine equation, divide 100 into two summands such that one is divisible by 7 and other by 11.

(c) Write a short note on Prime number theorem.

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$$2x + 5y \equiv 7 \pmod{13}.$$

(c) Use Fermat's theorem to verify that 17 divides $11^{104} + 1$.

(d) Find the remainder when $2(26!)$ is divided by 29.

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Prove $f(n) = \Sigma_{d|n}\mu(d)F(n/d) = \Sigma_{d|n}\mu(n/d)F(d)$.

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(d) Prove that $\varnothing(2^n - 1)$ is a multiple of n for any $n > 1$.

4. (a) For any positive integer n, prove

$\varnothing(n) = n\Sigma_{d|n}\mu(d)/d$.

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