Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **IV**

Unique Paper Code : **32351402**

Name of the Paper : **C9 - Riemann Integration and Series of Functions**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Find the upper and lower Darboux integrals for $f(x) = x^2 + 2$ on the interval $[0,1]$. Is f integrable on $[0,1]$? Justify.

   Let $g: [1,5] \to R$ be defined as

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 3, & \text{if } x \text{ is irrational.} \end{cases}$$

   Find the upper and lower sums, $U(g)$ and $L(g)$ respectively, on $[1,5]$. Is g integrable on $[1,5]$?

   Prove that $\left| \int_{-2\pi/3}^{2\pi/3} \left(\frac{3x}{4}\right)^2 \cos^3(x^5 + 3e^x) dx \right| \le \frac{\pi^3}{9}$ clearly stating the results used.

2. Show that the $Sgn\ x$, the Signum function, is integrable on $[-a, a]$ for any $a > 0$.

   Using Fundamental theorem of Calculus I, show that

$$\frac{1}{2}\int_2^3 x^{\frac{n}{2}}\, dx = \frac{3^{\frac{n+2}{2}} - 2^{\frac{n+2}{2}}}{n+2}.$$

   Calculate $\lim_{h\to 0} \frac{1}{h}\int_1^{1+h} e^{(3t^2+t)} dt$. State the theorem used.

3. Define improper integral of type II. When does it converge? When is it said to diverge?

   Examine the convergence of $\int_2^{+\infty} \frac{dx}{(x-1)^2}$ and $\int_{-\infty}^{0} e^{-x}\, dx$.

   Show that the improper integral $\int_0^1 x^{a+1}(1-x)^b dx$ converges if and only if $a > -2$ and $b > -1$.

4. Define $f_n : [0,1] \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} 2n - 2n^2 x & \text{if } \ 0 \le x \le 1/n \\ 0 & \text{if } \ 1/n < x \le 1. \end{cases}$$

Show that the pointwise limit does not exist on $[0,1]$. Further if we modify and define $f_n : [0,1] \to \mathbb{R}$ as

$$f_{n(x)} = \begin{cases} 0 & \text{if} \quad x = 0 \\ -n^2(2x - 2/n) & \text{if} \quad 0 < x \le 1/n \\ 0 & \text{if} \quad 1/n \ \le x \le 1 \end{cases}$$

then find the pointwise limit f of (f_n) on $[0,1]$. Show that f_n and f are integrable on $[0,1]$ but $lim_{n\to\infty} \int_0^1 f_n \ne \int_0^1 f$. Hence discuss the uniform convergence of (f_n) on $[0,1]$.

Let $f_n(x) = \frac{nx}{1+n^2x^2}$. Compute the pointwise limt $f(x) = lim_{n\to\infty} f_n(x)$. Show that the sequence (f_n) does not converge uniformly to f on $[-1,1]$ but does converge uniformly on $[1,\infty)$.

5. State a sufficient condition for the uniform convergence of a series of functions. Using this prove that $\sum_{n=1}^{\infty} \frac{e^{-2x}}{3n^2+2n}$ is uniformly convergent on $[0,\infty)$. Also show that it is a continuous function on $[0,\infty)$.

Examine the pointwise convergence of the series $\sum f_n$, where $f_n(x) = \frac{x^n}{2x^n+5}$, $x \ge 0$.

Show that $\sum_{n=1}^{\infty} \sin\left(\frac{x^2}{n^3}\right)$ is uniformly convergent on $(-1,1)$ but it is not uniformly convergent on $\mathbb{R}$.

6. Find the radius of convergence and determine the exact interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{2^k}{k^2} x^k, \quad \sum_{k=0}^{\infty} 2^{-k} x^{4k}.$$

Given

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \quad \text{for } |x| < 1,$$

show that $\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1}$ and hence $\frac{x}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^k$. Also evaluate $\sum_{k=1}^{\infty} (-1)^{k+1} k \left(\frac{3}{4}\right)^k$.

Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **IV**

Unique Paper Code : **32351403**

Name of the Paper : **C10 - Ring Theory and Linear Algebra-I**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. If $P_n(\mathbb{R})$ denotes the space of all polynomials of degree n or less with coefficients in $\mathbb{R}$ and if $T: P_1(\mathbb{R}) \to P_2(\mathbb{R})$ is a linear transformation such that $T(x+1) = x^2 - 1$ and $T(x-1) = x^2 + x$, what is $T(7x+2)$? Show that $\{x+1, x-1\}$ is a basis of a $P_1(\mathbb{R})$.

Let a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $T(a,b,c) = (a+c, a+b+2c, 2a+b+3c)$. Find a basis for the null space of *T* and a basis for the range space of *T*. Verify dimension theorem. Is *T* one-one? Is *T* onto?

2. Let a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ be defined as $T(x,y) = (x-2y, 2x+y, x+y)$. Let $\beta = \{(1,-1),\ (0,1)\}$ and $\gamma = \{(1,1,0), (0,1,1), (1,-1,1)\}$ be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Find the matrix of T with respect to β and γ, that is $[T]_\beta^\gamma$. If β' and γ' denote the standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively then find $[T]_{\beta'}^{\gamma'}$. What is the relation between the two matrices?

3. Can the polynomial $6x^3 - 3x^2 + x + 2$ be expressed as a linear combination of the polynomials $x^3 - x^2 + 2x + 3$ and $2x^3 - 3x + 1$ in $P_3(\mathbb{R})$? Justify.

Determine if the set

$$\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix}\right\}$$

in $M_{2\times 2}(\mathbb{R})$ is linearly independent or linearly dependent, where $M_{2\times 2}(\mathbb{R})$ is the set of all 2×2 matrices over $\mathbb{R}$. Justify.

Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ be a subset of the vector space F^3. Prove that if $F = \mathbb{R}$, then *S* is linearly independent and if *F* has characteristic 2, then *S* is linearly dependent.

4. Determine if the polynomials $x^2 + x + 1$, $2x^2 + 3x - 1$ and $-x^2 - x + 5$ generate $P_2(\mathbb{R})$. Justify.

What is the standard basis in $P_n(F)$? For a fixed $a \in \mathbb{R}$, determine the dimension of the subspaces of $P_n(\mathbb{R})$ defined by $\{f \in P_n(\mathbb{R}): f(a) = 0\}$.

5. Show that $S = \{a + ib : a, b \in \mathbb{Z}, b \text{ is even}\}$ is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$. Further, with complete explanation, find the number of elements in $\mathbb{Z}[i]/\langle 3 + i \rangle$. Find all maximal ideals in $\mathbb{Z}_8$.

6. Show that the field $\mathbb{Z}_3[i]$ is ring isomorphic to the field $\frac{\mathbb{Z}_3[x]}{\langle x^2+1 \rangle}$.

Determine all ring homomorphism from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$.

Name of the Course	: **B.Sc. (Hons.) Mathematics CBCS**
Semester	: **VI**
Unique Paper Code	: **32351601**
Name of the Paper	: **C13 - Complex Analysis**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Let $S = \{z \in \mathbb{C} : |z| < 2\}$ and let T denotes the boundary of S. Find interior points, exterior points, boundary points and accumulation points of T. Does there exists a sequence (z_n) in T such that the series

$$\sum_{n=1}^{\infty} z_n$$

converges? Justify your answer. Expand the function $1/(2 - z)$ into the Maclaurin series valid in the disk S. If $f: S \to T$ is a function such that f is analytic everywhere in S, prove that f is constant throughout S. If $g: \mathbb{C} \to T$ is an entire function, prove that g is constant throughout the complex plane.

2. Show that the function

$$f(z) = ze^{-z}$$

is entire by verifying that the real and imaginary parts of f satisfy the Cauchy–Riemann equations at each point of the complex plane. What is the anti-derivative of f? If C is any contour extending from $z = 0$ to $z = i\,\pi$, find the value of the integral

$$\int_C f(z)dz.$$

Also, use the ML-inequality to prove that

$$\left| \int_C \frac{f(z)}{z^2 - 1} dz \right| \leq \frac{2\pi\sqrt{e}}{3}$$

where C is the positively oriented circle $|z| = 1/2$.

3. Consider the function $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = (Im\, z)^2$. Use the Cauchy–Reimann equations to determine the points where f is differentiable. Is f analytic at those points? Compute the integral

$$\int_C f(z)dz$$

where C is the boundary of the square $\{0 < x < 1\, \&\, 0 < y < 1\}$ in the counter clockwise direction.

4. Let C be the positively oriented circle $|z| = 1$. Use the Cauchy Integral Formula to evaluate

$$\int_C \frac{\cos z}{z} dz.$$

Deduce that

$$\int_0^{2\pi} \cos(\cos t) \cosh(\sin t)\, dt = 2\pi.$$

Use the extension of Cauchy Integral Formula to find the value of the integral

$$\int_C \frac{e^z \cos z}{z^4} dz.$$

5. Find the pair of complex numbers z_1 and z_2 such that

$$Log\,(z_1 z_2) \neq Log\, z_1 + Log\, z_2$$

where $Log\, z$ represents the principal value of $\log z$. Is $Log(1+i)(1-i) = Log(1+i) + Log(1-i)$? Justify your answer. Expand the functions $z^3 - 6z^2 + 7z - 3$ into a Taylor series about the point $z_0 = 1$. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{z}{(z-1)(z-2)}$$

and specify the regions in which those expansions are valid.

6. Determine whether $z_0 = 0$ is a pole, a removable singularity or an essential singular point of the function

$$f(z) = \frac{1}{1 - \cos z}$$

and

$$g(z) = z^3 e^{1/z^2}.$$

Also, determine the residue of f and g at z_0. Use the substitution $z = e^{it}$ and the Cauchy Residue Theorem to evaluate the integral

$$\int_0^{2\pi} \frac{4 \cos x}{5 - 4\cos x} dx.$$

Name of the Course	: **B.Sc. (Hons.) Mathematics CBCS**
Semester	: **VI**
Unique Paper Code	: **32357607**
Name of the Paper	: **DSE-3: Probability Theory and Statistics**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Let the probability density function of a continuous random variable X be given by $f(x) = cx(1-x)$ for $0 < x < 1$ and $f(x) = 0$ elsewhere. Find the value of the constant c. Further, find the value of mean, variance, moment generating function of $Y = 3X + 1$ and the probability that Y lies within 2 standard deviations of the mean of Y. Compute the cumulative distribution function of Y and use it to find $P[Y > 2.5]$.

2. Consider a random experiment of shooting a bullet inside a circular disc of radius r. Let a random variable X denote the distance of the bullet mark from the center of the disc along the radius of the disc passing through the bullet mark. Find the average distance X, median distance X, standard deviation of X and the moment generating function of X. Also find the probability that the bullet will hit (i) exactly at the center of the disc (ii) exactly within a concentric circular disc of radius $r/4$ and (iii) exactly on the boundary of the concentric circular disc of radius $r/4$.

3. Let X_1 and X_2 have the joint probability mass function $p(x_1, x_2)$ given by

(x_1, x_2)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
$p(x_1, x_2)$	1/18	3/18	4/18	3/18	6/18	1/18

and $p(x_1, x_2) = 0$ elsewhere. Find the marginal probability mass functions $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$ and the conditional means $E(X_2/x_1)$ and $E(X_1/x_2)$.

Further, give example of two random variables X and Y such that X and Y are dependent but the covariance between X and Y is 0.

4. Let X and Y have the joint probability density function given by

$$f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the joint moment generating function of X and Y and hence compute the coefficient of correlation between X and Y.

5. State Markov and Chebyshev's inequalities and explain their significance. Let X be a random variable with $E(X) = 3$ and $E(X^2) = 13$. Use the Chebyshev's inequality to determine a lower bound for $P(-2 < X < 8)$. Further, obtain an upper bound for $P[\,|X| \geq 1]$ using Chebyshev's inequality if X is a random variable having probability mass function given by

x	-1	0	1
$p(x)$	$1/8$	$6/8$	$1/8$

6. Let $X_1, X_2, \dots X_n$ be a sequence of independent Bernoulli variates such that

$$P(X_i = 1) = p, P(X_i = 0) = q, \; p + q = 1.$$

Let $X = (\sum Xi)/n, \; 1 \leq i \leq n.$ Verify Central Limit Theorem by showing that X tends to be Normal as $n \to \infty$. A customer care center is operating in such a manner that the number of customers attended on a particular day is a random variable with mean of 250 persons and a standard deviation of 20 persons. Use the Central Limit Theorem or otherwise, find the probability that the average (mean) number of customers attended in a random sample of 45 days is at least 255?

Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **VI**

Unique Paper Code : **32357609**

Name of the Paper : **DSE-4: Bio-Mathematics**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. The rate of increase of bacteria in a culture is proportional to the number present. The population multiplies by the factor n in the time interval T. Find the number of bacteria at time t when the initial population is p_0. The population is found to increase by 2570 bacteria from $t = 2$ to $t = 3$ and by 5615 bacteria from $t = 4$ to $t = 5$. Show that $p_0 = 2460$ approximately and that, when $T = 3, n$ is about 3.23.

2. Let $X(t)$ denote the prey per unit area and $Y(t)$ denote the predator per unit area. Write a simple model for the growth of the prey density, and draw the graph for its population density. Subsequently modify the model to include the effect of depletion of natural resources on the prey population. Now take into account the predator-prey interaction and develop a differential equation model for the interaction. Hence, find the equilibrium points for the model.

3. What does a singular point signify? Examine what fixed points arise in the phase plane for

$$\frac{d^2x}{dt^2} - 2b\frac{dx}{dt} + ax = 0,$$

where a and b are constants ($a \neq 0$).

4. Show that the iteration scheme

$$x_{n+1} = 1 - \beta x_n(1 - x_n)$$

has a stable fixed point $x_0 = 1$ for $\beta < 1$ and that $\beta = 1$ is a bifurcation point where the fixed point $x_0 = \frac{1}{\beta}$ appears. By taking $x_1 = 0.250$, do you observe period doubling in the system as β exceeds 3? Justify.

$S_1 \backslash S_0$	A	G	C	T
A	7	0	1	0
G	1	9	1	1
C	0	1	7	2
T	1	0	1	8

5. Explain matrix model of base substitution from ancestral sequence S_0 to descendent sequence S_1. Also show that the probabilities of each base in the ancestral sequence S_0 are transformed into the probabilities of each base in the descendent sequence S_1 after one time step. From the given frequency table of 40-base sequence calculate the probabilities after one time step.

6. Draw a single topologically distinct un-rooted bifurcating tree that could describe the relationship between 3 taxa. Draw all the three topologically distinct rooted bifurcating trees that could describe the relationship between 3 taxa. Also draw all the three topologically distinct un-rooted bifurcating trees that could describe the relationship between 4 taxa. From the given distance table, construct a rooted tree showing the relationship between S1, S2, S3 and S4 by UPGMA.

	S1	S2	S3	S4
S1		0.06	0.12	0.14
S2			0.14	0.12
S3				0.22

Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **VI**

Unique Paper Code : **32357611**

Name of the Paper : **DSE-4: Linear Programming and Theory of Games**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Find all the existing basic feasible solutions for the following system of equations

$$5x_1 + 4x_2 + x_3 + x_4 + 2x_5 = 8$$
$$x_1 + x_2 + 3x_3 + 2x_4 + 6x_5 = 12.$$

Are the basic feasible solutions non-degenerate? Justify your answer.

2. Solve the following linear programming problem using Two Phase method

$$\begin{aligned} &\text{Maximize } z = 4x_1 + 5x_2 - 3x_3 \\ &\text{subject to} \quad x_1 + x_2 + x_3 = 10 \\ &\qquad -x_1 + x_2 \leq -1 \\ &\qquad x_1 + 3x_2 + x_3 \leq 14 \\ &\qquad x_1, x_2 \geq 0, \; x_3 \text{ unrestricted.} \end{aligned}$$

3. Write the dual of the following problem and solve the dual graphically.

$$\begin{aligned} &\text{Minimize } z = 6x_1 + 2x_2 \\ &\text{subject to} \quad 2x_1 - x_2 \geq -1 \\ &\qquad -3x_1 - 2x_2 \leq -3 \\ &\qquad x_1, x_2 \geq 0. \end{aligned}$$

Utilize the theorems of duality to obtain the values of all the primal variables from the optimal dual solution. State the theorems used.

4. Write the equivalent primal and dual linear programming problems of the following game theory problem

Player B

Player A $\begin{bmatrix} 2 & 0 & 3 \\ 3 & 5 & -3 \\ 5 & 1 & -3 \end{bmatrix}$

Solve any one of the linear programming problems to obtain optimum payoffs and optimal strategies for both the players.

5. The following table gives cost of transporting goods from factories A, B, C and D to destinations E, F, G, H and I

Destinations

Factories

	E	F	G	H	I	Supply
A	8	10	12	17	15	**100**
B	15	13	18	11	9	**150**
C	14	20	6	10	13	**160**
D	13	19	7	6	12	**280**
Demand	**70**	**170**	**50**	**210**	**190**	

Find the optimal transportation cost and optimal allocation of goods from factories to destinations. Determine alternate optimal allocation of goods with the same transportation cost.

6. A certain equipment needs five repair jobs which have to be assigned to five mechanics. The estimated time (in hrs.) that each mechanic requires to complete the repair job is given in the following table

Mechanic	Job				
	J_1	J_2	J_3	J_4	J_5
M_1	**3**	**10**	**3**	**8**	**2**
M_2	**6**	**8**	**7**	**6**	**1**
M_3	**6**	**8**	**7**	**5**	**3**
M_4	**4**	**2**	**7**	**3**	**1**
M_5	**8**	**6**	**12**	**8**	**4**

Assuming that each mechanic can be assigned to only one job, determine how each job is assigned to a mechanic to minimize the time allotment. Explain each iteration.

Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **VI**

Unique Paper Code : **32357610**

Name of the Paper : **DSE-4: Number Theory**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. When Dr. Khurana cashed a cheque at his bank, the cashier mistook the number of *Paise* for the number of *Rupees* and vice-versa. After spending 2 Rupees and 50 *Paise*, he realized that he had twice the amount of the original cheque. Determine the smallest value for which the cheque could have been written.

2. Let p_1, p_2, and p_3 be distinct primes, and let $0 \leq a_1 < p_1, 0 \leq a_2 < p_2$ and $0 \leq a_3 < p_3$ be given positive integers. Show that there exists a positive integer x_0 such that it leaves remainder a_1, a_2 and a_3 when divided by p_1, p_2 and p_3 respectively.

3. Is the converse of the Wilson's theorem true? If yes, prove it, else give a counter example. Also use the Wilson's theorem to prove that if p is a prime and a is any integer then $p|(a^p + (p-1)!\,a)$ and $p|((p-1)!\,a^p + a)$.

4. For any positive integer k, let $d_k(n)$ denote the number of ordered k −tuples of positive integers $(a_1, a_2, \dots, a_k)$ such that $n = a_1 a_2 \dots a_k$. By showing that $d_k(n)$ is multiplicative or otherwise, prove that $d_k(n) = \underbrace{(u * u * \dots * u)}_{k-times}(n)$, where $u(n) = 1$ for all n.

5. Let $k \geq 3$, and a be any odd integer. Prove that the order of a is at most 2^{k-2} modulo 2^k.

6. Prove that there are infinitely many primes in each of the sets $A = \{4n + 1 \mid n \in \mathbb{N}\}$ and $B = \{4n - 1 \mid n \in \mathbb{N}\}$.

Name of the Course	: **B.Sc. (Hons.) Mathematics CBCS**
Semester	: **VI**
Unique Paper Code	: **32351602**
Name of the Paper	: **C14 - Ring Theory and Linear Algebra-II**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Suppose that $f(x) \in \mathbb{Z}_p[x]$ and is irreducible over $\mathbb{Z}_p$, where p is a prime. If $\deg f(x) = n$, find the number of elements in the field $\mathbb{Z}_p[x]/\langle f(x)\rangle$.

Show that the polynomial $x^2 + 2x + 3$ is irreducible over $\mathbb{Z}_5$ and use this to construct a field of order 25.

2. Let D be a Euclidean domain and d the associated function. If a and b are associates in D, then what is the relation between $d(a)$ and $d(b)$? Justify your answer.

Also prove that $2 + 3i$ and $2 - 3i$ are not associates in $\mathbb{Z}[i]$.

3. Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows

$$f_1(x, y, z) = x + y, \quad f_2(x, y, z) = x - 2y + z, \quad f_3(x, y, z) = 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual basis.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Show that A is diagonalizable and find a 2×2 invertible martix Q such that $Q^{-1}AQ$ is a diagonal matrix. Also compute A^n where n is a positive integer.

5. Let $V = \mathbb{R}^3$, $u = (1, 2, 2)$ and $W = \{(x, y, z) : x + y - 3z = 0\}$. Find the orthogonal projection of the given vector u on the given subspace W of the inner product space V.

6. Show that every self-adjoint operator on a finite-dimensional inner product space is normal. Is the converse true? Justify your answer.

Let V be an inner product space over $\mathbb{R}$ and let T be a normal operator on V. Then show that $T - 3I$ is normal, where I is the identity operator on V.