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Sr. No. of Question Paper : 1204 **F**

Unique Paper Code : 2352011201

Name of the Paper : Linear Algebra

Name of the Course : **B.Sc. (H) Mathematics**

Semester / Type : II / DSC

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator not allowed.

1. (a) If x and y are vectors in $\mathbb{R}^n$, then prove that $||x + y|| \leq ||x|| + ||y||$. Also, verily the same for the vectors $x = [-1, 4, 2, 0, -3]$ and $y = [2, 1, -4, -1, 0]$ in $\mathbb{R}^5$.

(b) Using the Gauss – Jordan method, find the complete solution set for the following homogeneous system of linear equations:

$$4x_1 - 8x_2 - 2x_3 = 0$$

$$3x_1 - 5x_2 - 2x_3 = 0$$

$$2x_1 - 8x_2 + x_3 = 0$$

(c) Define the rank of a matrix. Using rank, find whether the non-homogeneous linear system AX = B, where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 0 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

has a solution or not. If yes, find the solution.

2. (a) Consider the matrix :

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & -1 & 5 \\ -4 & -3 & 3 \end{pmatrix}$$

Determine whether the vector [4, 0, –3] is in the row space of A. If so, then express [4, 0, –3] as a linear combination of the rows of A.

(b) Consider the matrix :

$$A = \begin{pmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{pmatrix}$$

(i) Find the eigenvalue and the fundamental eigenvectors of A.

(ii) Is A diagonalizable? Justify your answer.

(c) Find the reduced row echelon form matrix B of the following matrix :

$$A = \begin{pmatrix} 1 & 2 & -2 & -11 \\ 2 & 4 & -1 & -10 \\ 3 & 6 & -4 & -25 \end{pmatrix}$$

and then give a sequence of row operations that convert B back to A.

3. (a) Let F_1 and F_2 be fields. Let $\mathcal{F}(F_1, F_2)$ denote the vector space of all functions from F_1 to F_2. A function $g \in \mathcal{F}(F_1, F_2)$ is called an even function if $g(-t) = g(t)$ for each $t \in F_1$ and is called an odd

function if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even functions in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$.

(b) Let W_1 and W_2 be subspaces of a vector space V.

(i) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(ii) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(c) (i) Let S_1 and S_2 are arbitrary subsets of a vector space V. Show that if $S_1 \subseteq S_2$ then $\text{span}(S_1) \subseteq \text{span}(S_2)$.

(ii) Let F be any field. Show that the vectors (1,1.0), (1,0,1) and (0,1,1) generate F^3.

4. (a) Define a linearly independent subset of a vector space V. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k, $(1 \leq k < n)$.

(b) Let V be a vector space and $\beta = \{u_1, u_2, ..., u_n\}$ be a subset of V. Prove that β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form $v = a_1u_1 + a_2u_2 + ... + a_nu_n$, for unique scalars $a_1, a_2, ..., a_n$.

(c) Let F be any field. Consider the following subspaces of F^5:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_2 = a_3 = a_4 = 0, a_1 + a_5 = 0\}$$

Find bases and dimension for the subspaces W_1, W_2 and $W_1 \cap W_2$.

5. (a) Let V and W be vector spaces over a field F, and let $T : V \to W$ be a linear transformation. If $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for V then prove that

$R(T) = \text{span}\ (T(\beta)) = \text{span}(\{T(v_1), T(v_2), ..., T(v_n)\}$

If T is one-to-one and onto then prove that $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W.

(b) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear,

$T(1,1) = (1, -2)$

$T(-1, 1) = (2, 3).$

What is $T(-1,5)$ and $T(x_1, x_2)$?

Find $[T]_\beta^\gamma$ if $\beta = \{(1,1), (-1,1)\}$ and $\gamma = \{(1,-2), (2,3)\}$.

(c) For the following linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5 \\ -2 & 3 & -13 \\ 3 & -3 & 15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

find bases for null space N(T) and range space R(T). Also, verify the dimension theorem.

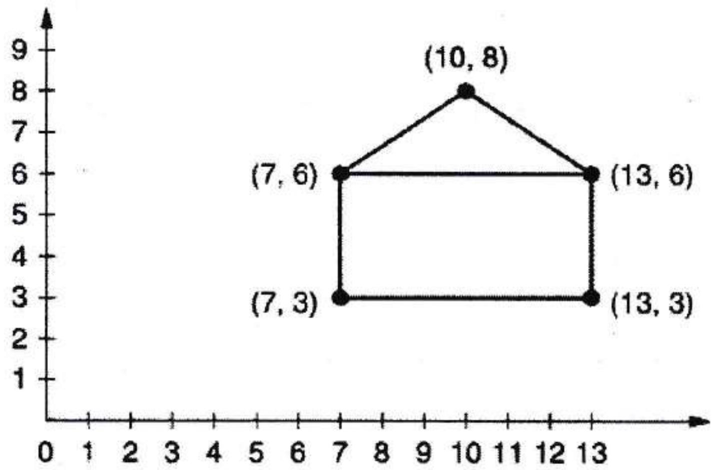
6\. (a) Let V and W be finite dimensional vector spaces over the same field F. Then, prove that V is isomorphic to W if and only if dim V = dim W. Are $M_{2\times2}(\mathbb{R})$ and $P_3(\mathbb{R})$ isomorphic? Justify your answer.

(b) Let V and W be vector spaces and let $T: V \to W$ be linear and invertible. Prove that $T^{-1}: W \to V$ is linear. For the linear transformation $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by :

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$$

determine whether T is invertible or not. Justify your answer.

(c) For the adjoining graphic, use homogenous co-ordinates to find the new vertices after performing scaling about (7,3) with scale factors of ½ in the x - direction and 3 in the y - direction.



Also, sketch the final figure that would result from this movement.

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Sr. No. of Question Paper : 1223

Unique Paper Code : 2352011202

Name of the Paper : CALCULUS

Name of the Course : **B.Sc. (H) Mathematics UGCF-2022**

Semester : II – DSC 5

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator is not allowed.

1. (a) State and prove the sequential criterion for the limit of a real valued function. (5)

   (b) Use $\in - \delta$ definition of limit to establish the following limit : (5)

$$\lim_{x \to 2} \frac{1}{1-x} = -1.$$

P.T.O.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as (5)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f has a limit only at $x = 0$.

(d) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If $\lim_{x \to c} f > 0$, then show that $f(x) > 0$ for all $x \in A \cap V_\delta(c)$, $x \neq c$. (5)

2. (a) If f is continuous at x_0 and g is continuous at $f(x_0)$ then prove that the composite function $g \circ f$ is continuous at x_0. (5)

(b) Let $f(x) = \frac{1}{x} \sin \frac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Show that f is discontinuous at 0. (5)

(c) State Intermediate Value Theorem. Prove that $xe^x = 1$ for some x in (0,1). (5)

(d) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$. (5)

3. (a) Prove that if a real valued function f is continuous on [a, b] then it is uniformly continuous on [a, b]. (5)

(b) Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on (a, ∞) for $a > 0$ but it is not uniformly continuous on $(0,1)$. (5)

(c) Let $f(x) = |x| + |x - 1|$, $x \in \mathbb{R}$. Draw the graph and give the set of points where it is not differentiable. Justify also. (5)

(d) Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$. (5)

4. (a) State and prove Mean Value Theorem. (5)

(b) State Intermediate Value Theorem for derivatives. Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$, $f(2) = 1$.

(i) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0,2)$.

(ii) Show that $f'(x) = \frac{1}{7}$ for some $x \in (0,2)$.

(5)

(c) Prove that $(\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. (5)

(d) Let f be defined on $\mathbb{R}$ and suppose that $|f(x) - f(y)| \le (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function. (5)

5. (a) Let f be differentiable function on an open interval (a, b). Then show that f is increasing on (a, b) if $f'(x) \ge 0$. (5)

(b) If $y = e^{\tan^{-1}x}$, prove that (5)

$(1 + x^2)y_{n+2} + (2(n + 1)x - 1)y_{n+1} + n(n + 1)y_n = 0.$

(c) If $y = \cos(m \sin^{-1} x)$, find $y_n(0)$. (5)

(d) Stating Taylor's theorem find Taylor series expansion of e^x. (5)

6. (a) Find $\lim_{x \to +\infty} \left[x - \ln\left(x^2 + 1\right)\right]$. (5)

(b) Determine the position and nature of the double points on the curve (5)

$x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0.$

(c) Sketch a graph of the rational function showing the horizontal, vertical and oblique asymptote (if any) of $y = \dfrac{x^2 - 2}{x}$. (5)

(d) Sketch the curve in polar coordinates of $r = \sin 2\theta$. (5)

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Sr. No. of Question Paper : 1242 **F**

Unique Paper Code : 2352011203

Name of the Paper : Ordinary Differential Equations

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester / Type : II / DSC

Duration : 3 Hours Maximum Marks : 90

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Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts of each question.

3. Each part carries **7.5** marks.

4. Use of non-programmable Scientific Calculator is allowed.

1. (a) Solve the initial value problem

$$(e^{2x}y^2 - 2x)\,dx + e^{2x}y\,dy = 0, \; y(0) = 2$$

(b) Solve

$$(2x + \tan y)\, dx + (x - x^2 \tan y)\, dy = 0$$

(c) Solve

(i) $(3x^2 + 4xy - 6)\, dy + (6xy + 2y^2 - 5)\, dx = 0$

(ii) $\frac{d^2y}{dx^2} = 2y\left(\frac{dy}{dx}\right)^3 = 2y$ by reducing the order.

2. (a) A certain rumor began to spread one day in a city with a population of 100,000. Within a week, 10,000 people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population has heard the rumor?

(b) The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident in a certain region has left the level of cobalt to be 100 times the acceptable level for habitation. How long will it be until the region is again habitable?

(c) A cake is removed from an oven at 210°F and left to cool at room temperature of 70°F. After 30 minutes, the temperature of the cake is 140°F. What will be its temperature after 40 minutes? When will the temperature be 100°F?

3. (a) Show that the solutions x, x^2, x log x of the third order differential equation

$$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$$

are linearly independent on $(0, \infty)$. Also find the particular solution satisfying the given initial condition.

$$y(1) = 3,\ y'(1) = 2,\ y''(1) = 1$$

(b) Solve the differential equation using the method of Variation of Parameters

$$\frac{d^2y}{dx^2} + 9y = \tan 3x$$

(c) Find the general solution of the differential equation using the method of undetermined Coefficients.

$$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}=4e^{-x}+3x^2$$

4. (a) Use the operator method to find the general solution of the following linear system

$$2\frac{dx}{dt}+\frac{dy}{dt}+x+5y=4t$$

$$\frac{dx}{dt}+\frac{dy}{dt}+2x+2y=2$$

(b) Solve the initial value problem. Assume $x > 0$.

$$x^2\frac{d^2y}{dx^2}-5x\frac{dy}{dx}+8y=2x^3, \quad y(2)=0, \ y'(2)=8$$

(c) A body with mass $m = \frac{1}{2}$ kg is attached to the end of a spring that is stretched 2m by a force of 16N. It is set in motion with initial position $x_0 = 1$m and initial velocity $v_0 = -5$m/s. Find the position function of the body as well as the amplitude, frequency and period of oscillation.

5. (a) Define the term Carrying Capacity. Derive the logistic equation

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right)$$

where K is the carrying capacity of the population. Also find the solution.

(b) The per-capita death rate for the fish is 0.5 fish per day per fish, and the per-capita birth rate is 1.0 fish per day per fish. Write a word equation describing the rate of change of the fish population. Hence obtain a differential equation for the number of fish. If the fish population at a given time is 240, 000, give an estimate of the number of fish born in one week.

(c) In an epidemic model where infected get recovered, the differential equation is of the form

$$\frac{dS}{dt} = -\beta SI, \ \frac{dI}{dt} = \beta SI - \gamma I$$

Use parameter values $\beta = 0.002$ and $\gamma = 0.4$, and assume that initially there is only one infective but there are 500 susceptibles. How many susceptibles never get infected, and what is the maximum number of infectives at any time? What happens as time progresses, if the initial number of susceptibles is doubled, $S(0) = 1000$? How many people were infected in total.

6. (a) A public bar opens at 6 p.m. and is rapidly filled with clients of whom the majority are smokers. The bar is equipped with ventilators that exchange the smoke-air mixture with fresh air. Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. The bar has a floor area of 20m by 15m, and a height of 4m. It is estimated that smoke enters the room at a constant

rate of 0.006 m^3/min, and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced. The problem is to establish a good time to leave the bar, that is, sometime before the concentration of carbon monoxide reaches the lethal limit. Starting from a word equation or a compartmental diagram, formulate the differential equation for the changing concentration of carbon monoxide in the bar over time. By solving the equation above, establish at what time the lethal limit will be reached.

(b) Find the equilibrium solution of the differential equation

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right)$$

And discuss the stability of equilibrium solution.

(c) Consider a disease where the infected get recovered. A model describing this is given by the differential equations

$$\frac{dS}{dt} = -\beta SI, \ \frac{dI}{dt} = \beta SI - \gamma I$$

P.T.O.

Use chain rule to find arelation between S and I. Obtain and sketch the phase-plane curves. Determine the direction of travel along the trajectories.