[This question paper contains 8 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : **1150** **A**

Unique Paper Code : 32351401_LOCF

Name of the Paper : BMATH-408 Partial Differential Equation

Name of the Course : **CBCS B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **four** questions.
3. **All** questions carry equal marks.

SECTION – I

Attempt any **two** parts out of the following.

Marks of each part are indicated.

1. (a) Define the following with one example each : (6)

(i) Quasi-linear first order partial differential equation (PDE).

(ii) Semi-linear first order PDE.

(iii) Linear first order PDE.

State whether the following first order PDE is quasi-linear, semi-linear, linear or non-linear :

$$\left(xy^2\right)u_x - \left(yx^2\right)u_y = u^2(x^2 - y^2)$$

Justify.

(b) Solve the Cauchy problem (6)

$$uu_x + u_y = 1$$

such that $u(s, 0) = 0, x(s, 0) = 2s^2,$

$$y(s, 0) = 2s, s > 0.$$

(c) Obtain the solution of the pde (6)

$$x\left(y^2 + u\right)u_x - y\left(x^2 + u\right)u_y = (x^2 - y^2)u\,,$$

with the data u(x,y) = 1 on x + y = 0.

(d) Apply $\sqrt{u} = v$ and $v(x,y) = f(x) + g(y)$ to solve

$$x^4 u_x^2 + y^2 u_y^2 = 4u \ . \qquad (6)$$

2. Attempt any **two** parts out of the following :

(a) Apply the method of separation of variables u(x,y) = f(x)g(y) to solve

$$y^2 u_x^2 + x^2 u_y^2 = (xyu)^2$$

such that $u(x,0) = 3e^{\frac{x^2}{4}}$. (6.5)

(b) Find the solution of the equation (6.5)

$$yu_x - 2xyu_y = 2xu$$

with the condition $u(0,y) = y^3$.

(c) Reduce into canonical form and solve for the general solution (6.5)

$$u_x - yu_y - u = 1.$$

(d) Derive the one-dimensional heat equation :

$$u_t = \kappa u_{xx},$$

where κ is a constant. (6.5)

SECTION – II

3. Attempt any **two** parts out of the following :

(a) Find the characteristics and reduce the equation

$u_{xx} - (\sec h^4 x)u_{yy} = 0$ into canonical form. (6)

(b) Find the characteristics and reduce the equation

$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} + xyu_x + y^2 u_y = 0$

into canonical form. (6)

(c) Transform the equation $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$ to the form $v_{\xi\eta} = cv$, c=constant, by introducing the new variable $v = ue^{-(a\zeta + b\eta)}$, where a and b are undetermined coefficients. (6)

(d) Use the polar co-ordinates r and θ (x=r cosθ, y=r sinθ) to transform the Laplace equation $u_{xx} + u_{yy} = 0$ into polar form. (6)

4. Attempt any **two** parts out of the following :

(a) Find the D'Alembert solution of the Cauchy problem for one dimensional wave equation given by

$$u_{tt} - c^2 u_{xx} = 0, x \in R, t > 0$$
$$u(x,0) = f(x), x \in R,$$
$$u_t(x,0) = g(x), x \in R.$$
(6.5)

(b) Solve (6.5)

$$y^3 u_{xx} - y u_{yy} + u_y = 0,$$
$$u(x,y) = f(x) \text{ on } x + \frac{y^2}{2} = 4 \text{ for } 2 \le x \le 4,$$
$$u(x,y) = g(x) \text{ on } x - \frac{y^2}{2} = 0 \text{ for } 0 \le x \le 2,$$
$$\text{with } f(2) = g(2).$$

(c) Determine the solution of initial boundary value problem

$$u_{tt} = 16 u_{xx}, \quad 0 < x < \infty, t > 0$$
$$u(x,0) = \sin x, \quad 0 \le x < \infty,$$
$$u_t(x,0) = x^2, \quad 0 \le x < \infty,$$
$$u(0,t) = 0, \quad t \ge 0.$$
(6.5)

(d) Determine the solution of initial boundary value problem (6.5)

$$u_{tt} = 9u_{xx}, \quad 0 < x, \infty, t > 0,$$
$$u(x,0) = 0, \quad 0 \le x < \infty,$$
$$u_t(x,0) = x^3, \quad 0 \le x < \infty$$
$$u_x(0,t) = 0, \quad t \ge 0.$$

SECTION - III

5. Attempt any **two** parts out of the following :

(a) Determine the solution of the initial boundary-value problem by method of separation of variables

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \ t > 0$$
$$u(x,0) = \begin{cases} hx/a, & 0 \le x \le a \\ h(l-x)/(l-a), & a \le x \le l \end{cases},$$
$$u_t(x,0) = 0, \qquad 0 \le x \le l,$$
$$u(0,t) = 0 = u(l,t) = 0 \qquad t \ge 0$$

(6.5)

(b) Obtain the solution of IBVP (6.5)

$$u_t = u_{xx}, \qquad 0<x<2, \ t>0,$$
$$u(x,0)=x, \ 0 \le x \le 2,$$
$$u(0,t)=0, \quad u_x(2,t)=1, \qquad t \ge 0,$$

(c) Determine the solution of the initial-value problem (6.5)

$$\begin{aligned} &u_{tt} = c^2 u_{xx} + x^2, \\ &u(x,0) = x, \quad 0 \le x \le 1, \\ &u_t(x,0) = 0, \quad 0 \le x \le 1, \\ &u(0,t) = 0, u(1,t) = 0, \quad t > 0. \end{aligned}$$

(d) Determine the solution of the initial-value problem (6.5)

$$\begin{aligned} &u_t = k u_{xx}, \quad 0 < x < 1, t > 0, \\ &u(x,0) = x(1-x), \quad 0 \le x \le 1 \\ &u(0,t) = t, \quad u(1,t) = \sin t, \quad t > 0. \end{aligned}$$

6. Attempt any **two** parts out of the following :

(a) Determine the solution of the initial boundary-value problem by method of separation of variables (6)

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, 0 < x < a, t > 0 \\ &u(x,0) = 0, \ 0 \le x \le a, \\ &u_t(x,0) = \begin{cases} \dfrac{v_0}{a} x, & 0 \le x \le a \\ v_0 (l-x)/(l-a), & a \le x \le l \end{cases} \\ &u(0,t) = 0 = u(a,t) = 0, \quad t \ge 0. \end{aligned}$$

P.T.O.

(b) Find the temperature distribution in a rod of length l. The faces are insulated, and the initial temperature distribution is given by x (l - x). (6)

(c) Establish the validity of the formal solution of the initial boundary – value problem (6)

$$u_t = ku_{xx}, \quad 0 \le x \le l, \; t > 0,$$
$$u(x,0) = f(x), \qquad 0 \le x \le l,$$
$$u(0,t) = 0, \qquad t > 0,$$
$$u_x(1,t) = 0, \qquad t > 0.$$

(d) Prove the uniqueness of the solution of the problem : (6)

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \; t > 0,$$
$$u(x,0) = f(x), \; 0 \le x \le l,$$
$$u_t(x,0) = g(x), 0 \le x \le l,$$
$$u(0,t) = u(0,t) = 0, \; t > 0.$$

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1377 **A**

Unique Paper Code : 32351402

Name of the Paper : BMATH-409; Riemann Integration and Series of Functions

Name of the Course : **B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 hours + 30 minutes Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Let f be a bounded function on [a, b]. Define integrability of f on [a, b] in the sense of Riemann. (6)

P.T.O.

(b) Prove that every continuous function on [a, b] is integrable. Discuss about the integrability of discontinuous functions. (6)

(c) Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that f is integrable on [–1, 1], Show that $\left|\int_{-1}^{1} f(t)dt\right| \leq 2$. (6)

(d) Let f(x) = x for rational x; and f(x) = 0 for irrational x. Calculate the upper and lower Darboux integrals of f on the interval [0, b]. Is f integrable on [0, b]? (6)

2. (a) State Fundamental Theorem of Calculus II. Use it to calculate

$$\lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}dt. \quad (6.5)$$

(b) Let f be defined as (6.5)

$$f(t)=\begin{cases}0, & t<0\\ t, & 0\leq t\leq 1\\ 4, & t>1\end{cases}$$

(i) Determine the function $F(x)=\int_0^x f(t)dt$.

(ii) Sketch F. Where is F continuous?

(iii) Where is F differentiable? Calculate F' at points of differentiability.

(c) State and prove Intermediate value theorem for Integral Calculus. Give an example to show that condition of continuity of the function cannot be relaxed. (6.5)

(d) Let f be a continuous function on $\mathbb{R}$. Define

$$G(x)=\int_0^{\sin x} f(t)dt \text{ for } x \in \mathbb{R}.$$

Show that G is differentiable on $\mathbb{R}$ and compute G'. (6.5)

3. (a) Let $\beta(p, q)$ (where $p, q > 0$) denotes the beta function, show that

$$\beta(p,q)=\int_{0^+}^{\infty}\frac{u^{p-1}}{(1+u)^{p+q}}du=\int_{0^+}^{1}\frac{v^{p-1}+v^{q-1}}{(1+v)^{p+q}}dv. \quad (6)$$

(b) Determine the convergence and divergence of the following improper integrals

(i) $\int_0^1 \frac{dx}{x(\ln x)^2}$

(ii) $\int_{-1}^{\infty} \frac{dx}{x^2+4x+6}$ (6)

(c) Define Improper Integral of type II.

Show that the improper integral $\int_1^{\infty} \frac{dx}{x^p}$ converges iff $p > 1$. (6)

(d) Show that the improper integral $\int_{\pi}^{\infty} \frac{\text{Sin } x}{x} dx$ is convergent but doesn't converge absolutely.

(6)

4. (a) Let $< f_n >$ be a sequence of integrable functions on $[a,b]$ and suppose that $< f_n >$ converges uniformly on [a, b] to f. Show that is f is integrable.

(6)

(b) Define

(i) pointwise convergence of sequence of functions

(ii) uniform convergence of a sequence of functions

(iii) If $A \subseteq \mathbb{R}$ and $\emptyset: A \to \mathbb{R}$ then define uniform norm of $\emptyset$ on A. (6)

(c) (i) Discuss the pointwise and uniform convergence of $f_n(x) = \frac{x}{n}$ for $x \in \mathbb{R}, n \in N$.

(ii) Show that the sequence $< f_n >$ where $f_n(x) = \frac{n}{x+n}, x \geq 0$ is uniformly convergent in any finite interval. (6)

(d) (i) Show that the sequence $< f_n >$ where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ uniformly convergent on $[0, \pi]$.

(ii) Discuss the pointwise and uniform convergence of the sequence $g_n(x) = x^n$ for $x \in \mathbb{R}, n \in N$. (6)

P.T.O.

5. (a) (i) State and prove the Cauchy Criteria for uniform convergence of series. (3.5)

(ii) Show that the series $\sum_{0}^{\infty}(1-x)x^{n}$ is not uniformly convergent on [0, 1] (3)

(b) Show that $\sum \frac{(-1)^{n}}{n^{p}} \frac{x^{2n}}{\left(1+x^{2n}\right)}$ converges absolutely and uniformly for all values of x if $p > 1$.

(c) Is the sequence $< f_{n} >$ where $f_{n} = \frac{\sin(nx+n)}{n}$, uniformly convergent on $\mathbb{R}$? Justify. (6.5)

(d) If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in N$ and $\sum f_n$ converges to f uniformly on D then prove that f is continuous on D. (6.5)

6. (a) Find the radius of convergence and exact interval of convergence of the following power series :

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} (x+1)^n$

(ii) $\sum_{n=0}^{\infty} x^{n!}$ (6.5)

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Show that the function f is differentiable on $(-R, R)$ and

$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$. (6.5)

(c) Show that

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ----$

for $|x| < 1$

(ii) $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} -----.$ (6.5)

(d) Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. If $0 < R_1 < R$, show that the power series converges uniformly on $[-R_1, R_1]$. Also, show that the sum function $f(x)$ is continuous on the interval $(-R, R)$. (6.5)

[This question paper contains 6 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1395 **A**

Unique Paper Code : 32351403

Name of the Paper : BMATH-410 – Ring Theory and Linear Algebra – I

Name of the Course : **CBCS (LOCF) B.Sc. (H) Mathematics**

Semester : IV

Duration : 3.30 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. All questions are compulsory.

1. (a) Define zero divisors in a ring. Let R be the set of all real valued functions defined for all real numbers under function addition and multiplication. Determine all zero divisors of R. (6½)

(b) What is nilpotent element? If a and b are nilpotent elements of a commutative ring, show that a + b is also nilpotent. Give an example to show that this may fail if the ring R is not commutative. (6½)

(c) Let R be a commutative ring with unity. Prove that U(R), the set of all units of R, form a group under multiplication of R. (6½)

(d) Determine all subrings of $\mathbb{Z}$, the set of integers. (6½)

2. (a) Define centre of a ring. Prove that centre of a ring R is a subring of R. (6)

(b) Suppose R is a ring with $a^2 = a$, for all $a \in R$. Show that R is a commutative ring. (6)

(c) Show that any finite field has order p^n, where p is prime. (6)

(d) Let R be a ring with unity 1. Prove that if 1 has infinite order under addition, then CharR = 0, and if 1 has order n under addition, then CharR = n. (6)

3. (a) Let R be a commutative ring with unity and let A be an ideal of R. Then show that R/A is a field if and only if A is maximal ideal. (6½)

(b) Prove that $I = \langle 2+2i \rangle$ is not prime ideal of $\mathbb{Z}[i]$. How many elements are in $\frac{\mathbb{Z}[i]}{I}$? What is the characteristic of $\frac{\mathbb{Z}[i]}{I}$? (6½)

(c) In $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients, let $I = \{f(x) \in \mathbb{Z}[x] \mid f(0) = 0\}$. Prove that I is not a maximal ideal. (6½)

(d) Let $\mathbb{R}[x]$ denote the ring of polynomials with real coefficients and let $\langle x^2+1 \rangle$ denote the principal ideal generated by x^2+1. Then show that

$$\frac{\mathbb{R}[x]}{\langle x^2+1\rangle} = \{g(x) + \langle x^2+1\rangle \mid g(x) \in \mathbb{R}[x]\} = \{ax+b+\langle x^2+1\rangle \mid a, b \in \mathbb{R}\}$$

(6½)

4. (a) If R is a ring with unity and the characteristic of R is $n > 0$, then show that R contains a subring isomorphic to $\mathbb{Z}_n$ and if the characteristic of R is 0 then R contains a subring isomorphic to $\mathbb{Z}$. (6)

(b) Determine all ring homomorphism from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$. (6)

(c) Let n be an integer with decimal representation $a_k a_{k-1} \cdots a_1 a_0$. Prove that n is divisible by 11 if and only if $a_0 - a_1 + a_2 - \cdots + (-1)^k a_k$ is divisible by 11. (6)

(d) Show that a homomorphism from a field onto a ring with more than one element must be an isomorphism. (6)

5. (a) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 + W_2$ is a smallest subspace of V that contains both W_1 and W_2. (6)

(b) For the following polynomials in $P_3(\mathbb{R})$, determine whether the first polynomial can be expressed as linear combination of other two. (6)

$$\{x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3\}.$$

(c) Let $S = \{u_1, u_2, ..., u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in span(\{u_1, u_2, ..., u_k\})$ for some $k\ (1 \leq k < n)$. (6)

(d) Let $W_1 = \{(a, b, 0) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}$ and $W_2 = \{(0, b, c) \in \mathbb{R}^3 : b, c \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Determine $\dim(W_1)$, $\dim(W_2)$, $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$. Hence deduce that $W_1 + W_2 = \mathbb{R}^3$. Is $\mathbb{R}^3 = W_1 \oplus W_2$? (6)

6. (a) Let V and W be finite-dimensional vector spaces having ordered bases β and γ respectively, and let $T : V \to W$ be linear. Then for each $u \in V$, show

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta.$$ (6½)

(b) Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be linear transformation defined by

$$T(a + bx + cx^2) = (a - c) + (a - c)x + (b - a)x^2 + (c - b)x^3.$$

Find null space N(T) and range space R(T). Also verify Rank-Nullity Theorem. (6½)

(c) For the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & -4 \\ 1 & -2 & 2 \end{bmatrix}$ and ordered basis $\beta = \{(1, 1, 0), (0, 1, 1), (1, 2, 2)\}$, find $[L_A]_\beta$. Also find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$. (6½)

(d) Let V and W be vector spaces and let $T: V \to W$ be linear. Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W. (6½)

[This question paper contains 6 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1114 **A**

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. 3. **All** questions are compulsory.
3. Attempt **two** parts from each question.

1. (a) Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi$ is mapped by the transformation $w = z^2$ and $w = z^3$. (6)

   (b) (i) Find the limit of the function $f(z) = \frac{(\bar{z})^2}{z}$ as z tends to 0.

P.T.O.

(ii) Show that $\lim_{z \to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i} = 2\sqrt{3}i.$ (3+3=6)

(c) Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \bar{z}^2/z & when\ \ z \neq 0 \\ 0 & when\ \ z = 0 \end{cases}$$

Verify that the Cauchy-Riemann equations are satisfied at the origin z = (0,0). (6)

(d) If $\lim_{z \to z_0} f(z) = F$ and $\lim_{z \to z_0} g(z) = G$, prove that

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{F}{G}$ if $G \neq 0$. (6)

2. (a) Find the values of z such that

(i) $e^z = 1 + \sqrt{3}i$, (ii) $e^{(2z-1)} = 1$. (3.5+3=6.5)

(b) Show that the roots of the equation $cosz = 2$ are $z = 2n\pi + icosh^{-1}2$ $(n = 0, \pm 1, \pm 2, \ldots)$, Then express them in the form $z = 2n\pi \pm i\ln(2+\sqrt{3})$ $(n = 0, \pm 1, \pm 2, \ldots)$.

(3.5+3=6.5)

(c) Show that (3.5+3=6.5)

(i) $\log(1+i)^2 = 2Log(1+i)$,

(ii) $\log(-1+\sqrt{3}i) = ln2 + 2\left(n+\frac{1}{3}\right)\pi i \ (n = 0, \pm1, \pm2, \ldots)$

(d) Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if

$z = n\pi \ (n = 0, \pm1, \pm2, \ldots)$. (6.5)

3. (a) (i) State mean value theorem of integrals.

Does it hold true for complex valued functions? Justify.

(ii) Evaluate $\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta$. (3+3=6)

(b) Parametrize the curves C_1 and C_2, where

C_1: Semicircular path from −1 to 1

C_2: Polygonal path from the vertices −1, −1+i, 1+i and 1

Evaluate $\int_{C_1} z\,dz$ and $\int_{C_2} z\,dz$. (3+3=6)

(c) For an arbitrary smooth curve C: $z = z(t)$, $a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integrals

(i) $\int_{z_1}^{z_2} z\,dz$ and

(ii) $\int_{z_1}^{z_2} dz$

depend only on the end points of C. (3+3=6)

(d) State ML inequality theorem. Use it to prove that $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$, where C denotes the line segment from z = i to z = 1. (2+4=6)

4. (a) A function f(z) is continuous on a domain D such that all the integrals of f(z) around closed contours lying entirely in D have the value zero. Prove that f(z) has an antiderivative throughout D. (6.5)

(b) State Cauchy Goursat theorem. Use it to evaluate the integrals

(i) $\int_C \frac{1}{z^2+2z+2} dz$, where C is the unit circle $|z| = 1$

(ii) $\int_C \frac{2z}{z^2+2} dz$, where C is the circle $|z| = 2$

(2.5+2+2=6.5)

(c) State and prove Cauchy Integral Formula.

(2+4.5=6.5)

(d) (i) State Liouville's theorem. Is the function $f(z) = \cos z$ bounded? Justify.

(ii) Is it true that 'If p(z) is a polynomial in z then the function $f(z) = 1/p(z)$ can never be an entire function'? Justify (4.5+2=6.5)

5. (a) If a series $\sum_{n=1}^{\infty} z_n$ of complex numbers converges then prove $\lim_{n\to\infty} z_n = 0$. Is the converse true? Justify. (6.5)

(b) Find the integral of $\int_C \frac{\cosh \pi z}{z^3+z}$ where C is the positively oriented circle $|z| = 2$. (6.5)

(c) Find the Taylor series representation for the function $f(z) = \frac{1}{z}$ about the point $z_0 = 2$. Hence prove that $\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$ for $|z - 2| < 2$. (6.5)

(d) If a series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges to f(z) at all points interior to some circle $|z - z_0| = R$, then prove that it is the Taylor series for the function f(z) in power of $z - z_0$. (6.5)

6. (a) For the given function $f(z) = \frac{z+1}{z^2+9}$ find the poles, order of poles and their corresponding residue. (6)

(b) Write the two Laurent Series in powers of z that represent the function $f(z) = \frac{1}{z+z^3}$ in certain domains and specify those domains. (6)

(c) Suppose that $z_n = x_n + iy_n$, $(n = 1,2,3,...)$ and $S = X + iY$. Then prove that

$\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$. (6)

(d) Define residue at infinity for a function f(z). If a function f(z) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then prove that

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$ (6)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1210 **A**

Unique Paper Code : 32357609

Name of the Paper : DSE-3: Bio-Mathematics

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. Use of Scientific Calculator is allowed.

1. (a) When the drug theophylline is administered for asthma, a concentration below 5 mg/ litre has little effect and undesirable side-effects appear if concentration exceeds 20 mg/ litre. For a body that weights W kg the concentration when Mmg is present is 2M/W mg/litre. If τ = 6 hours,

measures the rapidity at which concentration falls. Find the concentration at time t hours after the initial dose od D mg. If D = 500 mg and W = 70 kg show that the second dose is necessary after about 6 hours to prevent the concentration from becing ineffective. (6)

(b) Let X and Y represent the population of predator and prey respectively. In absence of predation, X decreases exponentially and Y follows a logistic growth. The rate at which prey is eaten is proportional to the product of densities of X and Y. Determine the rest states in this Volterra-Lotka Model of predator prey interaction. Also, find the appropriate relation between the constants of proportionality involved. (6)

(c) Determine the rest states in Belousov-Zhabotinskii reaction for Field-Noyes system given by

$$A + Y \rightarrow X$$

$$X + Y \rightarrow P$$

$$A + X \rightarrow 2X + Z$$

$$2X \rightarrow Q$$

$$Z \rightarrow fY,$$

where K_1, K_2, K_3, K_4, K_5 as rate constants respectively. (6)

(d) Describe the Hodgkin-Huxley model governing the nerve impulse transmission. (6)

2. (a) Discuss the nature of fixed point and give the equations of trajectories for the given system

$$\dot{x} = 5x - 5y$$
$$\dot{y} = 5x - 3y. \qquad (6\tfrac{1}{2})$$

(b) State the features required in the mathematical model of heartbeat cycle. Sate the heart beat model that represents the heart beat cycle with these features. (6½)

(c) What is a limit cycle? Give example. State Limit cycle criterion and Poincare-Bendixson theorem. (6½)

(d) Discuss the trajectories of $x'' - 4x' + 40x = 0$ in the phase plane by substituting

$$x = \rho \operatorname{Cos} \Phi, \ y = \rho \operatorname{Sin} \Phi. \qquad (6\tfrac{1}{2})$$

3. (a) Write the mathematical model of the Nerve Impulse Transmission due to FitzHugh and Nagumo for the Space Clamped Case. Draw the Nullclines when the total membrane current $I(t) > 0$ and when $I(t) < 0$. (6)

P.T.O.

(b) Show that the following system has limit cycle.

$$\frac{du}{dt} = u(1-u)(u-a) - w + I(t),$$

$$\frac{dw}{dt} = bu - \gamma w, \qquad 0 < a < 1,\ b > 0,\ \gamma \geq 0. \qquad (6)$$

(c) What is a control parameter? Show that μ is a control parameter for the following system.

$$\dot{x} = 5x + (3-\mu)y + 5x^3$$
$$\dot{y} = x + 5y + x^3 \qquad (6)$$

(d) Describe a full phase plane analysis for the heart beat equations

$$\varepsilon\frac{dx}{dt} = -(x^3 - Tx + b), \qquad T > 0$$

$$\frac{db}{dt} = (x - x_0) + (x_0 - x_1)u$$

By appropriately defining u, the control variable associated with the pacemaker, x is the muscle fibre length, b is the control variable and t is the time, (x_0, b_0) is the rest state and x_1 corresponds to the systolic state. (6)

4. (a) Show that the nonlinear conservative system

$$\dot{x} = y$$
$$\dot{y} = \mu \, Sin \, x - x, \quad \mu \geq 0,$$

has one equilibrium point for $0 \leq \mu < 1$ and three for $\mu S > 1$. Also discuss the nature of equilibrium points. (6½)

(b) Define :

(i) Bifurcation

(ii) Bifurcation Point

Make the sketches for Pitchfork bifurcation, Saddle-node bifurcation and Hopf bifurcation. (6½)

(c) Solve the ordinary differential equation

$$\frac{dy}{dt} = -\gamma y + u, \quad 0 < \gamma < 1/4.$$

To obtain the period T of the periodic state y, which characterizes the pacemaker. Assume that $0 \leq y \leq 1$, such that when y = 1, the pacemaker fires and when y = 0, it jumps back. (6½)

(d) Sketch the trajectories of the following system

$$\dot{x} = y$$

$$\dot{y} = \frac{1}{2}(1 - x^2)$$ (6½)

5. (a) Derive the formula for the Jukes-Cantor distance (d_{JC}) given that all the diagonal entries of Jukes-Cantor matrix M^t are $\frac{1}{4} + \frac{3}{4}\left(1 - \frac{4}{3}\alpha\right)^t$, where a is the mutation rate. Compute the Jukes-Cantor distance $d_{JC}(S_0, S_1)$ to 4 decimal digits, from the following 40 base table :

$S_1 \backslash S_0$	A	G	C	T
A	7	0	1	1
G	1	9	2	0
C	0	2	7	2
T	1	0	1	6

(6½)

(b) Describe when two trees are considered to be topologically similar. Draw all topologically distinct un-rooted bifurcation trees that could describe the relationship between 3 taxa and 4 taxa. (6½)

(c) In mice, an allele A for agouti- or gray-brown grizzled fur is dominant over the allele a, which determines a non-agouti color. If an Aa×Aa cross produces 6 offsprings, then compute the probabilities that :

(i) Exactly 4 of 6 offspring have agouti fur.

(ii) More than half of 6 offspring have agouti fur. (6½)

(d) Write a short note on Mendel's genetic theory. (6½)

6. (a) Define bifurcation tree and unrooted tree with examples of each. Draw the three topologically distinct rooted bifurcating trees that could describe the relationship between 3 taxa. (6)

(b) From the given distance table of four sequences S_1, S_2, S_3 and S_4 of DNA, construct a rooted tree showing the relationship between S_1, S_2, S_3 and S_4 by UPGMA

	S_1	S_2	S_3	S_4
S_1		1.2	0.9	1.7
S_2			1.1	1.9
S_3				1.6

(6)

(c) Explain Kimura 2-parameter and 3-parameter models along with their corresponding distance formulas. Write the expression of the log-det distance between S_0 and S_1. (6)

(d) If D and d denote the alleles for tall and dwarf plant and if W and w denote the alleles for round and wrinkled seed, then create a Punnett square for a DdWw × ddWw cross pea plant and compute the probability of a tall plant with round seeds. (6)

[This question paper contains 10 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1211 **A**

Unique Paper Code : 32357614

Name of the Paper : DSE-3 MATHEMATICAL FINANCE

Name of the Course : **B.Sc. (Hons) Mathematics CBCS (LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

3. **All** questions are compulsory and carry equal marks.

4. Use of Scientific calculator, Basic calculator and Normal distribution tables all are allowed.

1. (a) Explain Duration of a zero-coupon bond. A 4-year bond with a yield of 10% (continuously compounded) pays a 9% coupon at the end of each year.

P.T.O.

(i) What is the bond's price?

(ii) Use duration to calculate the effect on the bond's price of a 0.3% decrease in its yield?

(You can use the exponential values: $e^x = 0.9048$, 0.8187, 0.7408, and 0.6703 for $x = -0.1, -0.2, -0.3$, and -0.4, respectively)

(b) Explain Continuous Compounding. Suppose R_c denotes rate of interest with continuous compounding and R_m denotes equivalent rate with compounding m times per annum. Find the relation between R_c and R_m.

(c) An investor receives ₹ 1100 in one year in return for an investment of ₹ 1000 now. Calculate the percentage return per annum with :

(i) Annual compounding

(ii) Semi-annual compounding

(iii) Continuous compounding.

(You can use: $\ln(1.1) = 0.953$)

(d) Define Bond Yield and Par Yield. Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5% and 7% respectively. What is the 2-year par yield? (You can use the exponential values: $e^x = 0.9753, 0.9418, 0.9071, 0.8694$ for $x = -0.025, -0.06, -0.0975, -0.14$, respectively.)

2. (a) Explain Hedging. A United States company expects to pay 1 million Canadian dollars in 6 months. Explain how the exchange rate risk can be hedged using

(i) A Forward Contract

(ii) An Option.

(b) (i) What is the difference between the over-the counter market and the exchange-traded market?

(ii) An investor enters a short forward contract to sell 175,000 British pounds for US dollars at an exchange rate of 1.900 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is 2.420?

(c) A 1-year forward contract on a non-dividend paying stock is entered into when the stock price is ₹ 40, and the risk-free rate of interest is 10% per annum with continuous compounding. What is the forward price? Justify using no arbitrage arguments. ($e^{0.1} = 1.1052$)

(d) (i) A trader writes an October call option with a strike price of ₹ 35. The price of the option is ₹ 6. Under what circumstances does the trader make a gain,

(ii) Suppose that you own 6,000 shares worth ₹ 75 each. How can put options be used to provide an insurance against a decline in the value of the holding over the next 4 months?

3. (a) Draw the diagrams illustrating the effect of changes in stock price, strike price, and expiration date on European call and put option prices when

$S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.

(b) Derive the put-call parity for European options on a non-dividend-paying stock. Use put-call parity to derive the relationship between the delta of a European call and the delta of a European put on a non-dividend-paying stock.

(c) An investor sells a European call on a share for ₹ 4. The stock price is ₹ 47 and the strike price is ₹ 50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

(d) Define upper bound and lower bound for European options on a non-dividend- paying stock. What is a lower bound for the price of a 3-month European put option on a non-dividend-paying stock when the stock price is ₹38, the strike price is ₹40, and the risk-free interest rate is 10% per annum? Justify using no arbitrage arguments. ($_e$–0.04 = 0.9753)

4. (a) A 4-month European call option on a dividend-paying stock is currently selling for ₹ 50. The stock

price is ₹ 640, the strike price is ₹ 600, and a dividend of ₹ 8 is expected in 1 month. The risk-free interest rate is 12% per annum for all maturities. What opportunities are there for an arbitrageur? ($e^{-0.04} = 0.9608$)

(b) Consider a one-period binomial model where the stock can either go up from S_0 to S_0u ($u > 1$) or down from S_0 to S_0d ($d < 1$). Suppose we have an option with payoff f_u if the stock moves up and payoff f_d if the stock moves down. By considering a portfolio consisting of long position in Δ shares of stock and a short position in the option, find the price of the option. Explain how the price can be expressed as an expected payoff discounted by the risk-free interest rate.

(c) A stock price is currently ₹ 50. It is known that at the end of two months it will be either ₹ 53 or ₹ 48. The risk-free interest rate is 12% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of ₹ 49? Use no-arbitrage arguments. ($e^{0.02} = 1.0202$)

(d) Consider a two-period binomial model with current stock price S_0 = ₹ 100, the up factor u = 1.3, the down factor d = 0.8, T = 1 year and each period being of length six months. The risk-free interest rate is 5% per annum with continuous compounding. Construct the two-period binomial tree for the stock. Find the price of an American put option with strike K = ₹ 95 and maturity T = 1 year. ($e^{-0.025}$ = 0.9753)

5. (a) Stock price in the Black-Scholes model satisfies

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

where $\phi(m, v)$ denotes a normal distribution with mean m and variance v. Find $Var[S_T]$.

(b) What is the price of a European put option on a non-dividend-paying stock when the stock price is ₹ 69, the strike price is ₹ 70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

(You can use exponential values: $e^{-0.0144} = 0.9857$, $e^{-0.025} = 0.9753$)

(c) Let V be a lognormal random variable with ω being the standard deviation of ln V. Prove that

$$E[\max(V-K, 0)] = E(V)N(d_1) - KN(d_2)$$

where

$$d_1 = \frac{\ln\left[\frac{E(V)}{K}\right] + \frac{\omega^2}{2}}{\omega}, \quad d_2 = \frac{\ln\left[\frac{E(V)}{K}\right] - \frac{\omega^2}{2}}{\omega}$$

and E denotes the expected value. Use this result to derive the Black-Scholes formula for the price of a European call option on a non-dividend paying stock.

(d) A stock price is currently ₹ 50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in 2 years? Calculate the mean and standard deviation of the distribution. ($e^{0.18}$ =1.1972)

6. (a) Discuss gamma of a portfolio of options and calculate the gamma of a European call option on a non-dividend-paying stock where the stock price is ₹ 49, the strike price is ₹ 50, the risk-free

interest rate is 5% per annum and the time to maturity is 20 weeks, and the stock price volatility is 30% per annum. (ln(49/50) = –0.0202)

(b) What is the relationship between delta, theta and gamma of an option? Show by substituting for various terms in this relationship that it is true for a single European put option on a non-dividend-paying stock.

(c) Find the payoff from a bear spread created using put options. Also draw the profit diagram corresponding to this trading strategy.

(d) Companies X wishes to borrow US dollars at a fixed interest rate. Company Y wishes to borrow Indians rupees at a fixed rate of interest. The amounts required by the two companies are roughly the same at the current exchange rate. The companies have been quoted the following interest rates, which have been adjusted for the impact of taxes :

	Rupees	Dollars
Company X	9.6%	6.0%
Company Y	11.1%	6.4%

Design a swap that will net a bank, acting as intermediary, 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

[This question paper contains 8 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1299 **A**

Unique Paper Code : 32357616

Name of the Paper : DSE-4 Linear Programming and Applications

Name of the Course : **CBCS (LOCF) – B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions carry equal marks.

1. (a) Solve the following Linear Programming Problem by Graphical Method :

P.T.O.

Minimize $3x + 2y$

subject to $5x + y \geq 10$

$x + y \geq 6$

$x + 4y \geq 12$

$x \geq 0, y \geq 0$.

(b) Define a Convex Set. Show that the set S defined as :

$$S = \{(x,y) \mid x^2 + y^2 \leq 4\} \text{ is a Convex Set.}$$

(c) Find all basic feasible solutions of the equations:

$$x_1 + x_2 + 2x_3 + 3x_4 = 12$$

$$x_2 + 2x_3 + x_4 = 8$$

(d) Prove that to every basic feasible solution of the Linear Programming Problem:

$$\text{Minimize } z = cx$$

$$\text{subject to } Ax = b,\ x \geq 0$$

there corresponds an extreme point of the feasible region.

2. (a) Let us consider the following Linear Programming Problem:

$$\text{Minimize } z = cx$$

$$\text{subject to } Ax = b,\ x \geq 0$$

Let $(x_B, 0)$ be a basic feasible solution corresponding to a basis B having an a_j with $z_j - c_j > 0$ and all corresponding enteries $y_{ij} \leq 0$, then show that Linear Programming Problem has an unbounded solution.

(b) Let $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ be a feasible solution to the system of equations:

$$x_1 + 4x_2 - x_3 = 5$$

$$2x_1 + 3x_2 + x_3 = 18$$

Is this a basic feasible solution? If not, reduce it to two different basic feasible solutions.

(c) Using Simplex method, find the solution of the following Linear Programming Problem:

$$\begin{aligned} \text{Minimize} \quad & x_1 - 3x_2 + 2x_3 \\ \text{subject to} \quad & 3x_1 - x_2 + 2x_3 \leq 7 \\ & 2x_1 - 4x_2 \geq -12 \\ & -4x_1 + 3x_2 + 8x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(d) Solve the following Linear Programming Problem by Big-M method:

$$\begin{aligned} \text{Maximize} \quad & x_1 - 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 5x_3 = 40 \\ & x_1 + 2x_2 - 3x_3 \geq 22 \\ & 3x_1 + x_2 + 2x_3 = 30 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

3. (a) Solve the following Linear Programming Problem by Two Phase Method :

$$\begin{aligned} \text{Maximize} \quad & x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \geq 4 \\ & --2x_1 + 3x_2 - x_3 \leq 2 \\ & x_2 - 2x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(b) Find the solution of given system of equations using Simplex Method:

$$3x_1 - 2x_2 = 8$$

$$x_1 + 2x_2 = 4$$

Also find the inverse of A where $A=\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$.

(c) Using Simplex method, find the solution of the following Linear Programming Problem :

Maximize $2x_1 + x_2$

subject to $x_1 - x_2 \leq 10$

$2x_1 - x_2 \leq 40$

$x_1 \geq 0,\ x_2 \geq 0.$

(d) Find the optimal solution of the Assignment Problem with the following cost matrix :

Job \ Machines	I	II	III	IV	V	VI
A	4	8	5	4	6	9
B	8	3	8	4	11	7
C	9	5	7	9	8	7
D	10	9	5	6	9	9
E	5	11	9	10	10	9
F	9	5	7	10	8	7

4. (a) Find the Dual of following Linear Programming Problem :

Minimize $x_1 + x_2 + 3x_3$

subject to

$4x_1 + 8x_2 \geq 3$

$7x_2 + 4x_3 \leq 6$

$3x_1 - 2x_2 + 5x_3 = 7$

$x_1 \leq 0,\ x_2 \geq 0,\ x_3$ is unrestricted.

(b) State and prove the Weak Duality Theorem. Also show that if the objective function values corresponding to feasible solutions of the Primal and Dual Problem are equal then the respective solutions are optimal for the respective Problems.

(c) Using Complementary Slackness Theorem, find optimal solutions of the following Linear Programming Problem and its Dual:

Maximize $4x_1 + 3x_2$
subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 2 \\ x_1 - 2x_2 &\leq 3 \\ 2x_1 + 3x_2 &\leq 5 \\ x_1 + x_2 &\leq 2 \\ 3x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(d) For the following cost minimization Transportation Problem find initial basic feasible solutions by using North West Comer rule, Least Cost Method and Vogel's Approximation Method. Compare the three solutions (in terms of the cost):

Source \ Destination	A	B	C	D	E	Supply
I	15	15	16	17	15	24
II	18	19	16	20	15	38
III	16	15	22	17	20	43
Demand	27	12	32	17	17	

5. (a) Solve the following cost minimization Transportation Problem :

Origin \ Destinations	I	II	III	IV	Availability
A	10	11	10	13	30
B	12	12	11	10	50
C	13	11	14	18	20
Requirements	20	40	30	10	

(b) Four new machines are to be installed in a machine shop and there are five vacant places available. Each machine can be installed at to one and only one place. The cost of installation of each job on each place is given in table below. Find the Optimal Assignment. Also find which place remains vacant.

Machine \ Place	A	B	C	D	E
I	13	15	19	14	15
II	16	13	13	14	13
III	14	15	18	15	11
IV	18	12	16	12	10

(c) Define Maxmin and Minmax value for a Fair Game. Using Maxmin and Minmax Principle, find the saddle point, if exists, for the following pay – off matrix :

$$\text{Player 2} \overset{\text{Player 1}}{\begin{bmatrix} 1 & 3 & 6 \\ 5 & 4 & 2 \\ 2 & 1 & 3 \end{bmatrix}}.$$

(d) Convert the following Game Problem into a Linear Programming Problem for player A and player B and solve it by Simplex Method :

$$\text{Player A} \overset{\text{Player B}}{\begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 6 \end{bmatrix}}.$$

[This question paper contains 6 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1359 **A**

Unique Paper Code	:	32351602
Name of the Paper	:	BMATH614: Ring Theory and Linear Algebra II
Name of the Course	:	**B.Sc. (Hons.) Mathematics**
Semester	:	VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.

1. (a) (i) If D is an Integral domain, prove that D[x] is an integral domain.

   (ii) If R is a commutative ring, prove that the characteristic of R[x] is same as the characteristic of R.

   (b) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $Z_7[x]$. Compute the product f(x)g(x). Determine the quotient and the remainder upon dividing f(x) by g(x).

P.T.O.

(c) Let F be a field and let $I = \{a_n x^n + a_{n-1} x^{n-1} + \ldots.. + a_0 | a_i \in F$ and $f(1) = a_n + \ldots.\ a_0 = 0)$. Prove that I is an Ideal of F[x] and find a generator of I.

(d) Let R[x] denote the ring of polynomials with real coefficients. Then prove that $\frac{R[x]}{\langle x^2+1 \rangle}$ is isomorphic to the ring of complex numbers.

(3+3.5,6.5,6.5,6.5)

2. (a) (i) Let F be a field and $p(x) \in F[x]$ be irreducible over F. Prove that $\langle p(x) \rangle$ is a maximal ideal in F[x].

(ii) Show that, $\frac{Z_2[x]}{\langle x^3+x+1 \rangle}$ is a field with 8 elements.

(b) Determine which of the polynomials below are irreducible over Q.

(i) $3x^5 + 15x^4 - 20x^3 + 10x + 20$

(ii) $x^4 + x + 1$

(c) In integral domain $Z[\sqrt{-3}]$, prove that $1+\sqrt{-3}$ is irreducible but not prime.

(d) Define Euclidean domain. Prove that every Euclidean domain is a principal ideal domain.

(3+3,3+3,6,6)

3. (a) Let $V = P_1(R)$ and V^* denote the dual space of V. For $p(x) \in V$, define

$f_1, f_2 \in V^*$ by $f_1(p(x)) = \int_0^1 p(t)dt$ and $f_2(p(x)) = \int_0^2 p(t)dt$. Prove that $\{f_1, f_2\}$ is a basis for V^*

and find a basis for V for which it is the dual basis.

(b) Let W be a subspace of finite dimensional vector space V. Prove that

$\dim(W) + \dim(W^\circ) = \dim(V)$, where W° is annihilator of W.

(c) Let T be a linear operator on $M_{n\times n}(R)$ defined by $T(A) = A^t$. Show that ± 1 are the only eigenvalues of T. Find the eigenvectors corresponding to each eigenvalue. Also find bases for $M_{2\times 2}(R)$ consisting of eigenvectors of T.

P.T.O.

(d) Let T be a linear operator on R^3 defined by $T(a, b, c) = (3a + b, 3b + 4c, 4c)$. Show that T is digonalizable by finding a basis for R^3 consisting of eigen vectors of T. (6.5,6.5,6.5,6.5)

4. (a) Let T be a linear operator on finite dimensional vector space V and let W be the T-cyclic subspace of V generated by a non-zero vector $v \in V$. Let $k = \dim(W)$. Then prove that $\{v, T(v), \ldots\ldots\ldots, T^{k-1}(v)\}$ is basis for W.

(b) State Cayley Hamilton Theorem. Verify the theorem for linear operator $T: R^2 \to R^2$ defined by $T(a, b) = (a + 2b, -2a + b)$.

(c) Let T be a linear operator on R^3 defined by $T(a, b, c) = (3a - b, 2b, a - b + 2c)$. Find the characteristic polynomial and minimal polynomial of T.

(d) (i) Let T be an invertible linear operator. Prove that a scalar λ is an eigen value of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(ii) Prove that similar matrices have the same characteristic polynomial. (6,6,6,3+3)

5. (a) Show that in a complex inner product space V over field F. For x, y ∈ V, prove the following identities

(i) $\langle x, y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$ if $F = R$

(ii) $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \left\|x + i^k y\right\|^2$ if $F = C$, where $i^2 = -1$.

(b) Let V be an inner product space, and let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal subset of V. Prove the Bessel's Inequality :

$$\|x\|^2 \geq \sum_{i=1}^{n} \left|\langle x, v_i\rangle\right|^2 \text{ for any } x \in V.$$

Further prove that Bessel's Inequality is an equality if and only if $x \in \operatorname{span}(S)$.

(c) Let $V = P_2(R)$, with the inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)dt$$

and with the standard basis $\{1, x, x^2\}$. Use Gram-Scmidth process to obtain an orthonormal basis β of $P_2(R)$. Also, compute the Fourier coefficients of $h(x) = 1 + x$ relative to β.

(d) Find the minimal solution to the following system of linear equations

$x + 2y - z = 1$

$2x + 3y + z = 2$

$4x + 7y - z = 4$ (3+3.5,6.5,6.5,6.5)

6. (a) For the data $\{(-3, 9), (-2, 6), (0, 2), (1, 1)\}$, use the least squares approximation to find the best fit with a linear function and compute the error E.

(b) Let T be a linear operator on a finite dimensional inner product space V. Suppose that the characteristic polynomial of T splits. Then prove that there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular.

(c) (i) Let T be a linear operator on $\mathbb{C}^2$ defined by $T(a, b) = (2a + ib, a + 2b)$. Determine whether T is normal, self-adjoint, or neither.

(ii) For $z \in \mathbb{C}$, define $T_z: \mathbb{C} \to \mathbb{C}$ by $T_z(u) = zu$. Characterize those z for which T_z is normal, self adjoint, or unitary.

(d) Let U be a Unitary operator on an inner product space V and let W be a finite dimensional U-invariant subspace of V. Then, prove that

(i) $U(W) = W$

(ii) $W^\perp$ is U-invariant (6,6,3+3,3+3)

(3000)